United Kingdom Hydrographic Office

HM Nautical Almanac Office

NAO TECHNICAL NOTE

No. 77 2017 December

Time for a Specified Azimuth

by

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Summary

This technical note explores a complete method for calculating the time when the azimuth of a celestial body takes a particular value on a given date. It is assumed that the position of the observer is known and an ephemerides is available to calculate the Greenwich Hour Angle and declination of the body at any given time. Some examples are given, illustrating the ideas introduced.

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Contents

1 Introduction

The problem of calculating the time at which a celestial body will attain a particular altitude on a given date is well studied due to the cultural importance of rise, set and, for the Sun, twilight times. However, the related problem of finding the time at which a given azimuth is attained has been less well examined. *The Explanatory Supplement to the Astronomical Almanac* [1], [2] provides a method for solving this problem, however such a method is only applicable to cases where the latitude of the observer is greater than the declination of the celestial body.

This technical note explores a complete method for calculating the time when the azimuth, Z, of a body takes a particular value on a given date. It is assumed that the position of the observer is known and an ephemerides is available to calculate the Greenwich Hour Angle (GHA) and declination of the body at any given time.

The note begins by deriving equations for the hour angle, given the azimuth, the observer's latitude and the declination of the body at an initial estimate time, before analysing the conditions on its use. These conditions arise from differentiating two equations for the hour angle and comparing the signs of the results. Converting the hour angle into a time, given the observer's longitude and the GHA of the body at an initial estimate time is then discussed. These equations are then combined together in an iterative process to find the time at a specified azimuth and deduce the altitude at this time. A summary that expands the previously incomplete method given in *The Explanatory Supplement to the Astronomical Almanac* and some examples conclude these notes.

The azimuth convention throughout these notes is that North is located at 0°, East at 90°, South at 180° and West at 270°. The hour angle, t, is within the range $-180° \leq t \leq 180°$. Also here $|\phi| \leq 90°$ and $|\delta| \leq 90°$. Longitude is measured positively to the east.

2 Hour Angle Equations

To calculate the time at a given azimuth the first step is to determine the hour angle of the celestial body. To find this, the transformation of the local hour angle, t, and declination, δ, coordinate system to that of altitude, h, and azimuth, Z, is used. This involves a rotation of the reference frame through an angle $90 - \phi$ in the plane of the meridian, where ϕ is the geocentric latitude, followed by a rotation about the new vertical by 180° [1]. Thus,

$$\begin{bmatrix} \cos h \cos Z \\ \cos h \sin Z \\ \sin h \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin\phi & 0 & -\cos\phi \\ 0 & 1 & 0 \\ \cos\phi & 0 & \sin\phi \end{bmatrix} \begin{bmatrix} \cos\delta\cos t \\ \cos\delta\sin t \\ \sin\delta \end{bmatrix} \tag{1}$$

Hence

$$\cos h \cos Z = -\sin\phi\cos\delta\cos t + \cos\phi\sin\delta \tag{2}$$

$$\cos h \sin Z = -\cos\delta\sin t \tag{3}$$

and

$$\sin h = \cos\phi\cos\delta\cos t + \sin\phi\sin\delta \tag{4}$$

By combining equations (2) and (3) the altitude can be eliminated, giving

$$\tan Z = \frac{-\cos\delta\sin t}{-\sin\phi\cos\delta\cos t + \cos\phi\sin\delta} \tag{5}$$

which can be rearranged to find

$$\sin t = \tan Z \sin\phi\cos t - \tan Z\tan\delta\cos\phi \tag{6}$$

Let

$$a = \tan Z\sin\phi \tag{7}$$

$$b = -\cos\phi\tan\delta\tan Z \tag{8}$$

Substituting these into equation (6)

$$\sin t = a\cos t + b \tag{9}$$

As $\sin^2 x + \cos^2 x = 1$

$$a^2\cos^2 t + 2ab\cos t + b^2 + \cos^2 t - 1 = 0 \tag{10}$$

$$(a^2+1)\cos^2 t + 2ab\cos t + b^2 - 1 = 0 \tag{11}$$

Solving this quadratic equation gives

$$\cos t = \frac{-ab \pm \sqrt{1 + a^2 - b^2}}{1 + a^2} \tag{12}$$

To find t, the hour angle, use

$$t = \arccos(\cos t) \tag{13}$$

Note here that the function $\arccos(x)$ has the range of results $0° \leq x \leq 180°$. The actual hour angle, t, may not be in this range and a further condition needs to be applied to find it. If the result of $\arccos(\cos(t)) = x$ then

$$t = \begin{cases} x & \text{if } Z > 180° \\ -x & \text{if } Z < 180° \end{cases} \tag{14}$$

This condition arises because the local hour angle is always positive if the azimuth is westerly ($180° < Z < 360°$) and negative if the azimuth is easterly ($0° < Z < 180°$), and $\cos(x)$ is an even function. If $Z = 0°$ (directly North) or $Z = 180°$ (directly South) then t will either be $0°$ or $180°$ because the body will be passing over the observer's meridian, and hence a sign does not need to be applied.

Further conditions on the use of equation (12) to find the hour angle, t, are explored below. Firstly some special cases where there are no solutions or the equation cannot be used are analysed. Secondly, the use of the plus or minus sign within equation (12) is examined.

3 Special Cases

3.1 Cases Where There is No Solution

3.1.1 If $1 + a^2 - b^2 < 0$

If $1 + a^2 - b^2 < 0$ then there is no solution to equation (11). This means that for a given declination and latitude the given azimuth will never be attained for any hour angle. Therefore there is no time at which the given azimuth occurs.

3.1.2 Some Cases Where $|\phi| \leq |\delta|$

If $|\phi| \leq |\delta|$ then there will be a range of azimuths for which there is no solution to equation (6). This means that for a given declination and latitude the given azimuth will never be attained for any hour angle. Therefore there is no time at which the given azimuth occurs.

To find the range of azimuths for which this is the case equation (5) can be differentiated to find

$$\frac{dZ}{dt} = \frac{(\cos^2 \delta \sin \phi - \cos \phi \cos \delta \sin \delta \cos t) \cos^2 Z}{(\sin \phi \cos \delta \cos t - \cos \phi \sin \delta)^2} \tag{15}$$

The right hand side of equation (15) is equal to zero when

$$\cos t = \frac{\tan \phi}{\tan \delta} \tag{16}$$

giving the maximum and minimum values of t. Maximum and minimum values of the azimuth can be found from equation (5) (directly for $|\phi| < |\delta|$ and by rearranging and taking a limit for $|\phi| = |\delta|$). The correct quadrant can be deduced by examining the signs of the numerator and denominator. For $|\phi| = |\delta|$ the range is bounded by $Z = 90°$ and $Z = 270°$.

These values bound the range of azimuths for which there is no solution. If $\phi > \delta$ then the range is that which includes $Z = 360°$. If $\phi < \delta$ then the range is that which includes $Z = 180°$. If $\phi = \delta$ then the range in which there is no solution includes $Z = 360°$ if $\phi < 0$ or $Z = 180°$ if $\phi > 0$.

3.2 If $Z = 90°$ or $Z = 270°$

If $Z = 90°$ or $Z = 270°$ then equation (12) can not be used due to the nature of the function $\tan Z$ at these values of Z. Instead equation (2) can be considered, as in these cases $\cos Z = 0$ and it can therefore be simplified to

$$0 = -\sin \phi \cos \delta \cos t + \cos \phi \sin \delta \tag{17}$$

Rearranging this gives

$$\cos t = \frac{\tan\delta}{\tan\phi} = -\frac{b}{a} \tag{18}$$

Again, the condition that if the result of $\arccos(\cos(t)) = x$ then

$$t = \begin{cases} x & \text{if Z} > 180^\circ \\ -x & \text{if Z} < 180^\circ \end{cases} \tag{19}$$

must be applied.

4 Conditions on the Hour Angle Equation in Other Cases

If $1 + a^2 - b^2 \geq 0$ and $\cot Z \neq 0$, equation (12), along with the condition described below it, gives a solution for the hour angle, t, only when it is known whether the $+$ or $-$ sign before the square root should be used. Conditions which specify which to use can be derived by differentiating equation (12) with respect to the azimuth, Z,

$$\frac{d\cos t}{dZ} = \frac{d}{dZ}\left(\frac{-ab \pm \sqrt{1+a^2-b^2}}{1+a^2}\right) \tag{20}$$

If the $+$ sign in front of the square root in equation (20) is taken this can be expressed as

$$\begin{aligned} -\sin t\frac{dt}{dZ} = &-\frac{\left(\frac{da}{dZ}b + a\frac{db}{dZ}\right)}{(1+a^2)} + \frac{1}{2}\frac{\left(2a\frac{da}{dZ} - 2b\frac{db}{dZ}\right)}{(1+a^2)\sqrt{1+a^2-b^2}} \\ &- \frac{(-ab+\sqrt{1+a^2-b^2})2a\frac{da}{dZ}}{(1+a^2)^2} \end{aligned} \tag{21}$$

Using the following identities

$$b\frac{da}{dZ} = a\frac{db}{dZ} = \frac{ab}{\cos^2 Z\tan Z} \tag{22}$$

$$a\frac{da}{dZ} = \frac{a^2}{\cos^2 Z\tan Z} \tag{23}$$

$$b\frac{db}{dZ} = \frac{b^2}{\cos^2 Z\tan Z} \tag{24}$$

$$a\frac{da}{dZ} - b\frac{db}{dZ} = \frac{a^2-b^2}{\cos^2 Z\tan Z} \tag{25}$$

equation (21) can be simplified

$$-\sin t\frac{dt}{dZ} = \frac{[-2ab(1+a^2)\sqrt{1+a^2-b^2} + (a^2-b^2)(1+a^2) - 2a^2(-ab\sqrt{1+a^2-b^2} + 1 + a^2 - b^2)]}{(1+a^2)^2\sqrt{1+a^2-b^2}\cos^2 Z\tan Z} \tag{26}$$

$$-\sin t\frac{dt}{dZ} = \frac{-2ab\sqrt{1+a^2-b^2} - a^2 - a^4 - b^2 + a^2b^2}{(1+a^2)^2\sqrt{1+a^2-b^2}\cos^2 Z\tan Z} \tag{27}$$

$$-\sin t\frac{dt}{dZ} = \frac{-(a\sqrt{1+a^2-b^2} + b)^2}{(1+a^2)^2\sqrt{1+a^2-b^2}\cos^2 Z\tan Z} \tag{28}$$

If instead the $-$ sign in front of the square root in equation (20) is taken it simplifies to

$$-\sin t\frac{dt}{dZ} = \frac{(a\sqrt{1+a^2-b^2} - b)^2}{(1+a^2)^2\sqrt{1+a^2-b^2}\cos^2 Z\tan Z} \tag{29}$$

in a similar manner to when the $+$ sign is taken.

	If $\cos t = \frac{-ab+\sqrt{1+a^2-b^2}}{1+a^2}$	If $\cos t = \frac{-ab-\sqrt{1+a^2-b^2}}{1+a^2}$
$0 < Z < 90$	$-$	$+$
$90 < Z < 180$	$+$	$-$
$180 < Z < 270$	$-$	$+$
$270 < Z < 360$	$+$	$-$

Table 1: Sign of right hand side of equation (20).

In equation (28) the sign of the right hand side will be that of $-\tan Z$. In equation (29) the sign of the right hand side will be that of $\tan Z$. These signs are detailed in Table 1.

As the two sides of equation (20) must either be both positive, both negative or both equal to zero, the sign of its left hand side must be examined. Consider the three situations, $0° < t < 180°$, $-180° < t < 0°$ and $t = 0°$ or $t = 180°$.

4.1 If $0° < t < 180°$

If the hour angle, t, is within the range $0° < t < 180°$, then $\sin(t) > 0$ and $Z > 180°$. Hence

$$-\sin(t)\frac{dt}{dZ}\begin{cases} > 0 & \text{if } \frac{\mathrm{dt}}{\mathrm{dZ}} < 0 \\ < 0 & \text{if } \frac{\mathrm{dt}}{\mathrm{dZ}} > 0 \end{cases} \tag{30}$$

4.1.1 If Both Sides of Equation (20) are Negative

From condition (30), $\frac{dt}{dZ} > 0$. From Table 1, if the body is in the northern half of the sky ($270° < Z < 360°$) then the chosen sign in equation (12) must be $-$ and if it is in the southern half of the sky ($180° < Z < 270°$) then the chosen sign must be $+$.

4.1.2 If Both Sides of Equation (20) are Positive

From condition (30), $\frac{dt}{dZ} < 0$. From Table 1, if the body is in the northern half of the sky ($270° < Z < 360°$) then the chosen sign in equation (12) must be $+$ and if it is in the southern half of the sky ($180° < Z < 270°$) then the chosen sign must be $-$.

4.2 If $-180° < t < 0°$

If the hour angle, t, is within the range $-180° < t < 0°$, then $\sin(t) < 0$ and $Z < 180°$. Hence

$$-\sin(t)\frac{dt}{dZ}\begin{cases} < 0 & \text{if } \frac{\mathrm{dt}}{\mathrm{dZ}} < 0 \\ > 0 & \text{if } \frac{\mathrm{dt}}{\mathrm{dZ}} > 0 \end{cases} \tag{31}$$

4.2.1 If Both Sides of Equation (20) are Negative

From condition (31), $\frac{dt}{dZ} < 0$. From Table 1, if the body is in the northern half of the sky ($0° < Z < 90°$) then the chosen sign in equation (12) must be $+$ and if it is in the southern half of the sky ($90° < Z < 180°$) then the chosen sign must be $-$.

4.2.2 If Both Sides of Equation (20) are Positive

From condition (31), $\frac{dt}{dZ} > 0$. From Table 1, if the body is in the northern half of the sky ($0° < Z < 90°$) then the chosen sign in equation (12) must be $-$ and if it is in the southern half of the sky ($90° < Z < 180°$) then the chosen sign must be $+$.

4.3 If $t = 0°$ or $t = 180°$

If the hour angle, t, is equal to $0°$ or $180°$ then $Z = 0°$ or $Z = 180°$ and $\sin(t)$ is equal to zero. Hence both sides of equation (20) are equal to zero. The continuity of the above conditions shows that if $Z = 0°$ and $\frac{dt}{dZ} > 0$ then the $-$ sign is used and if $\frac{dt}{dZ} < 0$ then the $+$ sign is used. If $Z = 180°$ and $\frac{dt}{dZ} > 0$ then the $+$ sign is used and if $\frac{dt}{dZ} < 0$ then the $-$ sign is used.

4.4 If $\frac{dt}{dZ} = 0$

Differentiating equation (5) results in

$$\frac{dZ}{dt} = \frac{(\cos^2\delta\sin\phi - \cos\phi\cos\delta\sin\delta\cos t)\cos^2 Z}{(\sin\phi\cos\delta\cos t - \cos\phi\sin\delta)^2} \tag{32}$$

This differentiation is not valid for $Z = 90°$ or $Z = 270°$ due to the nature of $\tan Z$, however those values are dealt with in section 3.2 and not considered here. Rearranging,

$$\frac{dt}{dZ} = \frac{(\sin\phi\cos\delta\cos t - \cos\phi\sin\delta)^2}{(\cos^2\delta\sin\phi - \cos\phi\cos\delta\sin\delta\cos t)\cos^2 Z} \tag{33}$$

This implies that $\frac{dt}{dZ} = 0$ when $\cos t = \frac{\tan\delta}{\tan\phi} = -\frac{b}{a}$. Substituting this back into equation (5) implies that $Z = 90°$ or $Z = 270°$. As these two values have been excluded it is concluded that $\frac{dt}{dZ} \neq 0$ for any given latitude, declination and azimuth, excluding the values $Z = 90°$ or $Z = 270°$,

4.5 Summary of Conditions

Hence there are two conditions that determine if the $+$ or the $-$ sign before the square root must be chosen in equation (12) to find the hour angle, t. These are:

- The sign of $\frac{dt}{dZ}$.
- Whether the body is in the southern half of the sky ($90° < Z < 270°$) or in the northern half of the sky ($270° < Z \leq 360°$ or $0° \leq Z < 90°$).

In combination these conditions are

- If $\frac{dt}{dZ} > 0$ and $270° < Z \leq 360°$ or $0° \leq Z < 90°$, use the $-$ sign.
- If $\frac{dt}{dZ} < 0$ and $270° < Z \leq 360°$ or $0° \leq Z < 90°$, use the $+$ sign.
- If $\frac{dt}{dZ} > 0$ and $90° < Z < 270°$, use the $+$ sign.
- If $\frac{dt}{dZ} < 0$ and $90° < Z < 270°$, use the $-$ sign.

4.6 Finding the Sign of $\frac{dt}{dZ}$

The problem under consideration here is to find the time at a specified azimuth. While whether the body is in the northern or southern half of the sky can always be easily determined given an input azimuth, the sign of $\frac{dt}{dZ}$ can not always be resolved.

Equation (33) implies that the sign of $\frac{dt}{dZ}$ depends solely on the sign of $\cos\phi(\tan\phi - \tan\delta\cos t)$. Because $-90° \leq \phi \leq 90°$, $\cos\phi \geq 0°$ so the sign of $\frac{dt}{dZ}$ is the same as the sign of $\tan\phi - \tan\delta\cos t$.

4.6.1 If $|\phi| > |\delta|$

In this case the sign of $\frac{dt}{dZ}$ can be determined given the observer's latitude, ϕ, and the declination, δ, of the body.

- If $\phi > \delta$ then $\tan\phi - \tan\delta\cos t$ is positive and so $\frac{dt}{dZ}$ is positive.
- If $\phi < \delta$ then $\tan\phi - \tan\delta\cos t$ is negative and so $\frac{dt}{dZ}$ is negative.

4.6.2 If $|\phi| = |\delta|$

Again, in this case the sign of $\frac{dt}{dZ}$ can be determined given the observer's latitude, ϕ, and the declination, δ, of the body.

- If $\phi = \delta$ then $\tan\phi - \tan\delta\cos t$ can be simplified to $\tan\phi(1-\cos t)$. Equation (2) shows that the sign of $\cos Z$ is the same as that of $\sin\phi$ so a northern hemisphere latitude ($\frac{dt}{dZ} > 0$) will result in bodies only in the northern half of the sky, and a southern hemisphere latitude ($\frac{dt}{dZ} < 0$) will result in bodies only in the southern half of the sky. So, combining this with the conditions listed above, the $-$ sign before the square root in equation (12) should always be chosen.
- If $\phi = -\delta$ then $\tan\phi - \tan\delta\cos t$ can be simplified to $\tan\phi(1+\cos t)$. Equation (2) shows that the sign of $\cos Z$ is the same as that of $-\sin\phi$ so a northern hemisphere latitude ($\frac{dt}{dZ} > 0$) will result in bodies only in the southern half of the sky, and a southern hemisphere latitude ($\frac{dt}{dZ} < 0$) will result in bodies only in the northern half of the sky. So, combining this with the conditions listed above, the $+$ sign before the square root in equation (12) should always be chosen.

4.6.3 If $|\phi| < |\delta|$

In this case the sign of $\tan\phi - \tan\delta\cos t$ can not be determined as in section 4.6.1. As $-90^\circ \le \phi \le 90^\circ$ and $-90^\circ \le \delta \le 90^\circ$ the condition $|\phi| < |\delta|$ implies $\tan\phi < \tan\delta$ (because tan is, between -90° and 90°, a monotonically increasing, odd function). So the sign of $\tan\phi - \tan\delta\cos t$ will depend on the hour angle, t, and during a full cycle of $-180^\circ \le t \le 180^\circ$ there will be a continuous set of t where $\frac{dt}{dZ}$ is positive and a continuous set where $\frac{dt}{dZ}$ is negative.

Therefore there are two times at which a specified azimuth will occur, one for each of the signs of $\frac{dt}{dZ}$ and hence one for choosing the $+$ sign before the square root in equation (12) and one for choosing the $-$ sign. Without further information both times are equally valid answers to the problem of finding the time for a specified azimuth.

5 Converting Hour Angle to Time

The Local Hour Angle, $LHA = t$, can be related to the Greenwich Hour Angle GHA and the longitude, λ using the equation

$$LHA = GHA + \lambda \tag{34}$$

Here λ is negative to the west and positive to the east. As the $LHA = t$ is a function of the declination, δ, which itself is a function of time, UT, equation (34) can be rearranged to produce an iterative process

$$UT = UT_0 - (GHA + \lambda - t)/15 \tag{35}$$

where UT_0 is the initial estimate of the time. Equation (35) should be iterated, calculating t for each iteration using equation (12) and its conditions described above, until UT differs from UT_0 by less than the required precision. Multiples of 24 hours should be removed and care must be taken if the time goes over 24 hours into an adjacent day. If, during the iterative process, $\cos t > 1$ or $\cos t < -1$ then set $\cos t = 1$ or $\cos t = -1$ respectively. Choosing a range of different starting values of UT_0 will help find all possible solutions.

6 Altitude

Having found the time, UT, and therefore the precise declination, the calculated altitude can be found using equation (4). However to find the observed altitude refraction and, for the Moon, parallax must be applied. Firstly apply parallax for the Moon using an iterative process where

$$P_0 = \pi \cos h \tag{36}$$

and

$$P_{n+1} = \pi + \cos(h - P_n) \tag{37}$$

where P is the parallax, π is the horizontal parallax in radians and h is the calculated altitude. Iterate until $|P_n + 1 - P_n| \leq 0.0000024$. Then use

$$\text{Observerved altitude without refraction} = \text{Calculated altitude} - P \tag{38}$$

For the Sun and other celestial bodies $P = 0$. Refraction must then be applied using

$$\text{Observerved altitude} = \text{Calculated altitude} + \text{Refraction} \tag{39}$$

Refraction, R, can be calculated using the approximation [1]

$$R = \begin{cases} \dfrac{0.^\circ 5743 + 0.0705h + 0.00007h^2}{1 + 0.505h + 0.0845h^2} & \text{if h} < 15^\circ \\ 0.^\circ 01617 \tan(90^\circ - h) & \text{if h} \geq 15^\circ \end{cases} \tag{40}$$

7 Summary

Section 12.3.3.4 of *The Explanatory Supplement to the Astronomical Almanac* [1] is currently incomplete as the method that it describes to find the time for a specified azimuth is only applicable to cases where the latitude of the observer is greater than the declination of the celestial body. The following provides a summary of the method and conditions described above and serves as a replacement to the method described in section 12.3.3.4.

The problem is to calculate the time when the azimuth Z of a body takes a particular value. If $|\phi| > |\delta|$ then all azimuths are possible. If $|\phi| \leq |\delta|$, then there is range of azimuths for which solutions can be found. For $|\phi| < |\delta|$ the range is bounded by azimuths for which

$$\tan Z = \frac{-\cos\delta \sin t}{-\sin\phi \cos\delta \cos t + \cos\phi \sin\delta} \tag{41}$$

where

$$\cos t = \frac{\tan\phi}{\tan\delta} \tag{42}$$

The correct quadrant for $\tan Z$ can be deduced by examining the signs of the numerator and denominator. For $|\phi| = |\delta|$ the range is bounded by $Z = 90^\circ$ and $Z = 270^\circ$. Both these ranges include $Z = 180^\circ$ if $\phi > \delta$, or $Z = 360^\circ$ if $\phi < \delta$. If $\phi = \delta$ then ranges include $Z = 180^\circ$ if $\phi < 0$, or $Z = 360^\circ$ if $\phi > 0$.

If the azimuth is possible, the time UT is calculated iteratively from

$$UT = UT_0 - (GHA + \lambda \pm t)/15 \tag{43}$$

where the plus sign is used if $Z < 180^\circ$ and the minus sign is used otherwise. t is calculated from

$$\cos t = \frac{-ab \pm \sqrt{1 + a^2 - b^2}}{1 + a^2} \tag{44}$$

where $a = \tan Z \sin\phi$ and $b = -\cos\phi \tan\delta \tan Z$. If $|\phi| > |\delta|$ then there is one valid solution to this equation and the sign must be chosen to find the correct solution. In this case,

$$\text{if } \phi > \delta \text{ use the plus sign if } 90^\circ < Z < 270^\circ, \text{ otherwise use the minus sign;}$$
$$\text{if } \phi < \delta \text{ use the minus sign if } 90^\circ < Z < 270^\circ, \text{ otherwise use the plus sign.}$$

If $|\phi| = |\delta|$ then

if $\phi = \delta$ always use the minus sign;
if $\phi = -\delta$ always use the plus sign.

If $|\phi| < |\delta|$, then both solutions are valid. In the special cases $Z = 90°$ or $Z = 270°$ then

$$\cos t = -b/a = \tan\delta/\tan\phi \tag{45}$$

If, during the iterative process, $\cos t > 1$ or $\cos t < -1$ then set $\cos t = 1$ or $\cos t = -1$ respectively. If $1 + a^2 - b^2 < 0$ then there is no solution.

Iterate equation (43) until UT differs from UT_0 by less than $0^{\mathrm{h}}.008$. Choosing a range of different starting values of UT_0 will help find all possible solutions.

8 Examples

Example (1) Find the times when the Sun has an azimuth of 63° on 2016 April 17 from a location 45° East and 8° North. Also find the altitude at these times.

To begin the iterative process, take $UT_0 = 6^{\mathrm{h}}$. At this time $\delta =10\overset{\circ}{.}64$. As $|\phi| < |\delta|$ then there are two solutions. Taking the minus sign in equation (44) results in $t =144\overset{\circ}{.}10$, giving a time of 2223 from equation (43) (as $GHA =270\overset{\circ}{.}12$). Iterating this result gives $t =143\overset{\circ}{.}58$, giving a time of 2225 and a further iteration does not result in any change of time. At this time the calculated altitude can be found from equation (4) to be $-49\overset{\circ}{.}13$. As this is below the horizon it can not be observed.

Taking the plus sign in equation (44) results in $t =5\overset{\circ}{.}34$, giving a time of 0838 from equation (43). A further iteration does not result in any change of time. At this time the calculated altitude can be found from equation (4) to be $84\overset{\circ}{.}02$. The observed altitude also rounds to $84\overset{\circ}{.}02$.

Example (2) Find the times when Mars has an azimuth of 43° on 2016 August 14 from a location 104° West and 50° South. Also find the altitude at these times.

To begin the iterative process, take $UT_0 = 6^{\mathrm{h}}$. At this time $\delta =-23\overset{\circ}{.}89$. As $|\phi| > |\delta|$, $\phi < \delta$ and $Z < 90°$ there is one solution and the sign that must be taken in equation (44) is the plus sign. This results in $t =23\overset{\circ}{.}06$, giving a time of 0001 from equation (43) (as $GHA =170\overset{\circ}{.}76$). Iterating this result does not change the time. At this time the calculated altitude can be found from equation (4) to be $58\overset{\circ}{.}59$. The observed altitude is $58\overset{\circ}{.}31$.

This example illustrates how a different starting value of UT_0 can find an occurrence of a double phenomena. If $UT_0 = 23^{\mathrm{h}}$ is taken then $\delta =-23\overset{\circ}{.}95$. As $|\phi| > |\delta|$, $\phi < \delta$ and $Z < 90°$, again, there is one solution and the sign that must be taken in equation (44) is the plus sign. This results in $t =23\overset{\circ}{.}03$, giving a time of 2359 from equation (43) (as $GHA =66\overset{\circ}{.}11$). Iterating this result does not change the time. At this time the calculated altitude can be found from equation (4) to be $58\overset{\circ}{.}39$. The observed altitude is $58\overset{\circ}{.}40$.

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