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Continued Fractions Applied to Some Astronomical Problems

1. Continued fractions and their application to calendars

by

D.B. Taylor

Summary

This technical note is in two parts. The first part gives the basic theory of continued fractions. In particular, how to determine from a continued fraction the principal and intermediate convergents. Emphasis is given on their practical application and how they may be determined from a computer program. Some examples are given illustrating the ideas introduced. The second part uses continued fractions to identify astronomical periods used in forming calendars. The periods leading to rules used in the Gregorian and Islamic calendars are discussed and also the cycles used in ancient Greek calendars.

HM Nautical Almanac Office
UK Hydrographic Office
Admiralty Way, Taunton
TA1 2DN England

Telephone: +44 (0) 1823 337900
Facsimile: +44 (0) 1823 284077
Email: hmnao@ukho.gov.uk
WWW: http://astro.ukho.gov.uk/

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1. Continued fractions and application to calendars

D.B. Taylor

1. Introduction

Continued fractions, a topic in number theory in pure mathematics, has applications in astronomical problems. For the practical purposes for which we will use this theory, we can state the continued fraction for a positive real number t leads to a series of rational convergents which are *best approximations* to t and tend to this value in the limit.

The near periodic motion of orbits of bodies in the solar system such as planets around the Sun or satellites around a planet leads often to closely repeated configurations of two or more bodies. It is this near periodicity and possible closely repeated configurations which allow continued fractions to be useful in certain astronomical problems.

Some of the earliest uses in astronomy of recognising the near periodicity of bodies was in the design of calendars. Here the important periods are that of the Earth about the Sun and the Moon about the Earth; the first giving rise to the length of a solar year and the latter to the length of a lunar month. It is the incommensurability of these astronomical periods which leads to the difficulty in deriving calendars. A good description of the leading calendars used today can be found in Richards (2013), and for historical calendars see Fotheringham (1935). From continued fractions for these periods and in addition to their ratio we can explain some of the rules employed by calendar designers.

The Antikythera Mechanism is a scientific instrument made in Rhodes about 87 BC. It is a calendar computer made up of 32 bronze gears, including a differential gear; the gearwheels have different numbers of teeth and the gear ratio of specific gear trains can be calculated. Zeeman (1986) discusses this system of gears in detail and in particular uses continued fractions to demonstrate how one such gear ratio approximates the ratio of the sidereal year to the sidereal month.

Christiaan Huygens (1681,1682,1703) used continued fractions as best approximations in his development of a gear driven model of the solar system. His model showed the relative motions of the six known planets and separately the Moon around the Earth. He used convergents in the continued fraction development of the ratios of the periods of the planet's orbit to the Earth's year. This work is discussed and reanalysed in Chapter IV in Rockett and Szüsz (1992).

Newcomb (1882a) gave a theory of the recurrence of solar eclipses based on properties of the near 18-year eclipse cycle or Saros. He produced tables of eclipses from 700 BC to 2300 AD. The Saros is found from one of the convergents in the continued fraction for μ/μ', where μ is the mean motion of the Moon and μ' is the mean motion of the Sun, both quantities being measured from the node. For this convergent we find the time for 242 revolutions of the Moon relatively to either of the nodes is very nearly equal to the period of 19 revolutions of the Sun relatively to the same node.

Some of the major perturbations in the solar system result from the near commensurability of mean motions between two bodies. An example of this are perturbations on Jupiter and Saturn, which is discussed in Chapter XI in Brouwer and Clemence (1961). Letting n, n' be the mean motions of Jupiter and Saturn respectively, from the continued fraction for n'/n we obtain the convergents 1/2, 2/5, 29/72, 60/149, ... The second convergent leads to the well-known long-period inequality in the motions of these planets. The period is about 900 years and there results significant perturbations in the mean longitudes of these planets.

In transits of Venus and Mercury, the possible intervals in time of successive transits at either node can be predicted. This was demonstrated by Message (2004) using a transit limit formula and interpreting convergents in the continued fraction for n'/n, where n, n' are the mean motions of the Earth and planet respectively, to a movement of the conjunction line from an initial position. The transit limit used was derived from the eclipse limit formula of Chauvenet (1863), volume I p.436, suitably adapted for transits. Newcomb (1882b), using the concept of movable conjunction points in his theory for prediction of recurrence of eclipses, applied it to recurrence of transits of Mercury. In it he made use of the convergents in the continued fraction for the ratio of the period of Mercury divided by the period of the Earth.

From the above examples, briefly summarised here, we see the power of using continued fractions in a wide range of astronomical problems. This technical note is divided into two parts; the first giving the basic theory of continued fractions and the second to the use of them in formulating rules in some calendars.

In the basic theory, we describe the mathematical nomenclature used, terminology and define a simple way of generating convergents, suitable for coding in a computer program. Next we show how in addition to the convergents, henceforth called the principal convergents, we can also generate the intermediate convergents. We then define what we mean by best approximation to an irrational number. The emphasis here is the practical use of continued fractions and when a theorem or result is quoted a reference is given where the reader can find a proof. The reference we refer to here is the book *Continued Fractions* by Andrew M Rockett and Peter Szüsz. This section is concluded with some examples to illustrate the techniques introduced.

The section on calendars is not intended in any way to give the history of the calendar. The reader is referred, for example, to Richards (2013), Fotheringham (1935) and entry on the calendar in the *Encyclopaedia Britannica*, 1911 for a detailed discussion. Instead, this note attempts to explain the cycles used in some calendars originating from the astronomical periods of the tropical year and synodic month. Using the theory developed earlier we derive the continued fractions for the tropical year, the synodic month and synodic month divided by the tropical year. We then use these results to explain rules adopted in some well-known calendars.

In order to discuss these results fully it is necessary to include the results of recent work on the Gregorian calendar by Rockett and Szüsz (1986) and on the Metonic cycle by Zeeman (1986). Whilst a brief description of the relevant part of their work is given here the reader is referred to the original source for a deeper understanding.

This technical note is the first in a series in which various astronomical problems are discussed and which continued fractions play a key roll.

2. **Continued fractions**

2.1 **Basic ideas**

We can represent a number t as a *regular continued fraction*

$$t = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}} \tag{1}$$

in which a_0 is an integer and the (possibly infinite) sequence $a_1,\ a_2,\ a_3,\ a_4,\ \ldots$ consists of positive integers. The numbers $a_0,\ a_1,\ a_2,\ a_3,\ a_4,\ \ldots$ are called the *partial quotients*. If the continued fraction for t does not end then t is an irrational number.

Two forms of notation to write (1) more compactly are frequently used

$$t = a_0 + \frac{1}{a_1+}\ \frac{1}{a_2+}\ \frac{1}{a_3+}\ \frac{1}{a_4+}\ \cdots \tag{2}$$

and

$$t = [\,a_0;\ a_1,\ a_2,\ a_3,\ a_4,\ \ldots] \tag{3}$$

The latter form for the continued fraction will be used in this note.

Consider the rational functions formed from the continued fraction by truncating it at first a_0, then a_1, then a_2 etc. Thus

$$\begin{aligned} C_0 &= a_0 \\ C_1 &= a_0 + \frac{1}{a_1} = \frac{a_0\,a_1 + 1}{a_1} \\ C_2 &= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}} = \frac{a_0\,(a_1\,a_2 + 1) + a_2}{a_1\,a_2 + 1} \end{aligned}$$

and so on. The C_k's are the *convergents* of the continued fraction.

We can define two sequences $\{A_k\}_{k\geq -1}$ and $\{B_k\}_{k\geq -1}$ in terms of the a_k's by setting

$$\begin{aligned} &A_{-1} = 1, \quad A_0 = a_0 \quad \text{and} \quad B_{-1} = 0,\, B_0 = 1 \\ &A_{k+1} = a_{k+1}\, A_k + A_{k-1} \\ &B_{k+1} = a_{k+1}\, B_k + B_{k-1} \quad \text{for} \quad 0 \leq k \leq n-1 \end{aligned} \tag{4}$$

Then it can be shown

$$C_0 = \frac{A_0}{B_0}, \quad C_1 = \frac{A_1}{B_1}, \quad C_2 = \frac{A_2}{B_2}$$

and in general

$$C_k = \frac{A_k}{B_k}. \tag{5}$$

The A_k's and B_k's also have the property that

$$A_k\, B_{k-1} - A_{k-1}\, B_k = (-1)^{k-1} \quad \text{for} \quad k \geq 0 \tag{6}$$

It is a simple matter to calculate the convergents in a computer program by firstly establishing the partial quotients, then computing the A_k, B_k from (4) and hence convergents from (5). The property (6) can be used as a check of the validity of A_k, B_k from knowledge of A_{k-1}, B_{k-1}.

Suppose for simplicity $t > 1$ be irrational. Then we can determine the continued fraction for the reciprocal of t from the partial quotients for t in (1). We have

$$\frac{1}{t} = [\,0;\ a_0,\ a_1,\ a_2,\ a_3,\ \ldots] \tag{7}$$

and given the convergents $\{A_k/B_k\}_{k\geq -1}$ to t then if $\{\alpha_k/\beta_k\}_{k\geq -1}$ be the convergents to $1/t$ we have

$$\alpha_k = B_{k-1}, \quad \beta_k = A_{k-1}, \quad \text{for} \quad k = 0,\, 1,\, 2,\, \ldots \tag{8}$$

2.2 **Best approximations**

For the applications of continued fractions in astronomy we consider only regular continued fractions.

It can be shown that

$$t = \lim_{k\to\infty} \frac{A_k}{B_k} \tag{9}$$

exists. The convergents $\{A_k/B_k\}_{k\geq 0}$ approximate t alternately from below and above.

We define a rational number a/b a *best approximation* of the number t if

$$|b\,t - a| < |q\,t - p| \tag{10}$$

for any rational number p/q different from a/b with $0 < q \leq b$.

From the theory of continued fractions we have:

Theorem

Let t be irrational. Then A/B is a best approximation to t if and only if it is a convergent of t. (For proof see Rockett and Szüsz(1992) pp.26 -27).

The convergents A_k/B_k we term the *principal convergents*. From the partial quotients and sequences $\{A_k\}_{k\geq 0}$, $\{B_k\}_{k\geq 0}$ we can define quotients

$$\frac{A_k + c_{k+2}\, A_{k+1}}{B_k + c_{k+2}\, B_{k+1}}, \ c_{k+2} = 1,\ \ldots,\ a_{k+2} - 1,\ \text{and}\ a_{k+2} > 1 \tag{11}$$

These are termed *intermediate convergents* (also can be called *quasi* or *auxiliary convergents*). Note in (11) if we put $c_{k+2} = 0$ or a_{k+2} we obtain principal convergents A_k/B_k and A_{k+2}/B_{k+2} respectively.

The definition of best approximation described earlier by equation (10) is sometimes termed "a best approximation of the second kind". We can describe "a best approximation of the first kind" to be a fraction a/b (where $b > 0$) such that

$$|t - a/b| < |t - p/q| \tag{12}$$

for any fraction $p/q \neq a/b$ with $0 < q \leq b$.

A best approximation of the second kind is also of the first kind. It can be shown (see Rockett and Szüsz (1992) p.36) that every best approximation of the first kind is either a principal convergent or an intermediate convergent. The converse statement does not hold (see Ex.2 in section 2.3). We refer the reader to Rockett and Szüsz (1992) for further reading on this topic and in general on the theory of continued fractions.

2.3. **Examples of continued fractions**

Example 1

Consider the golden ratio. Let a line of length 1 be divided into two parts with the larger length being x. Letting g denote the "golden ratio" we have

$$g = \frac{1}{x} = \frac{x}{1-x}$$

Now

$$\frac{x}{1-x} = \frac{1}{\frac{1}{x} - 1} = \frac{1}{g-1}$$

therefore

$$g = \frac{1}{g-1}$$

or

$$g = 1 + \frac{1}{g} \tag{13}$$

From (13) we find

$$g = \frac{1+\sqrt{5}}{2} = 1{\cdot}6180\ 33\ldots$$

The continued fraction for g can thus be found as

$$[\,1;\ 1,\ 1,\ 1,\ 1,\ \ldots]$$

from (1), (3) or simply from (13). Thus the partial quotients are all 1 and from (4) and (5) we can find the principal convergents

$$1, \quad 2, \quad \frac{3}{2}, \quad \frac{5}{3}, \quad \frac{8}{5}, \quad \ldots$$

The sequence $\{B_k\}_{k\geq -1}$ for g is called the Fibonacci sequence. As the partial quotients are all 1 there are no intermediate convergents for g.

Example 2

Consider the continued fraction for $\sqrt{6}$.

$$\sqrt{6} = 2{\cdot}4494\ 89\ldots$$

We find the continued fraction for $\sqrt{6}$ is

$$\sqrt{6} = [\,2;\ 2,\ 4,\ 2,\ 4,\ \ldots]$$

The partial quotients are therefore

$$a_0 = 2,\ a_1 = 2,\ a_2 = 4,\ a_3 = 2,\ a_4 = 4,\ \ldots$$

and we obtain the principal convergents

$$2, \quad \frac{5}{2}, \quad \frac{22}{9}, \quad \frac{49}{20}, \quad \frac{218}{89}, \quad \ldots$$

that is in the terminology of section 2

$$\frac{A_0}{B_0}=\frac{2}{1},\quad \frac{A_1}{B_1}=\frac{5}{2},\quad \frac{A_2}{B_2}=\frac{22}{9},\quad \frac{A_3}{B_3}=\frac{49}{20},\quad \frac{A_4}{B_4}=\frac{218}{89},\quad \ldots$$

As there are a_i's greater than 1, from (11) we can compute the intermediate convergents.

$k=0$ $$\frac{A_0+c_2\,A_1}{B_0+c_2\,B_1},\qquad c_2=1,\;\ldots,\;a_2-1$$

or $$\frac{2+c_2\,5}{1+c_2\,2},\qquad c_2=1,\,2,\,3$$

giving $\frac{7}{3}$, $\frac{12}{5}$ and $\frac{17}{7}$.

$k=1$ $$\frac{A_1+c_3\,A_2}{B_1+c_3\,B_2},\qquad c_3=1,\;\ldots,\;a_3-1$$

or $$\frac{5+c_3\,22}{2+c_3\,9},\qquad c_3=1$$

giving $\frac{27}{11}$.

$k=2$ $$\frac{A_2+c_4\,A_3}{B_2+c_4\,B_3},\qquad c_4=1,\;\ldots,\;a_4-1$$

or $$\frac{22+c_4\,49}{9+c_4\,20},\qquad c_4=1,\,2,\,3$$

giving $\frac{71}{29}$, $\frac{120}{49}$, and $\frac{169}{69}$.

As we can compute further principal convergents obtained from calculating more partial quotients, so from these, further intermediate convergents can be computed.

We note whereas all principal convergents satisfy (10) none of the intermediate convergents do. All principal convergents will also satisfy (12) but now some of the intermediate convergents will do as well. From the intermediate convergents determined above 12/5, 17/7, 120/49, 169/69 will satisfy (12) but 7/3, 27/11 and 71/29 will not since

$$\left|\sqrt{6}-\frac{7}{3}\right| \not< \left|\sqrt{6}-\frac{5}{2}\right|$$
$$\left|\sqrt{6}-\frac{27}{11}\right| \not< \left|\sqrt{6}-\frac{22}{9}\right|$$
$$\left|\sqrt{6}-\frac{71}{29}\right| \not< \left|\sqrt{6}-\frac{49}{20}\right|$$

3. **Determination of astronomical cycles in some leading calendars**

As previously mentioned in the introduction it is not intended here in any way to give the history of the calendar. Instead, this note endeavours to describe the major astronomical cycles present in leading calendars based on

(i) the period the Earth orbits the Sun, the solar calendar

(ii) the period the Moon orbits the Earth, the lunar calendar, and

(iii) those which attempt to incorporate both periods (i) and (ii), the luni-solar calendar.

Examples of (i) and (ii) are the Gregorian and Islamic calendars respectively and of (iii) ancient Greek calendars. We show how continued fractions can be used to identify the astronomical cycles used in calendars of this type.

3.1 **Solar calendars**

From the *Explanatory Supplement to the Astronomical Almanac*, 2013 p.586 the mean tropical year taken from Laskar (1986) in days is

$$365{\cdot}2421\,8966\,98 - 0{\cdot}0000\,0615\,359\,T - 7{\cdot}29\times10^{-10}\,T^2 + 2{\cdot}64\times10^{-10}\,T^3 \tag{14}$$

with $T = (JD - 245\,1545{\cdot}0)/36525$ and JD is the Julian date.

Taking $T = 0$ i.e $JD = 245\,1545{\cdot}0$ in (14), we find for a time of noon on January 1,2000 the tropical year in days is $365{\cdot}2421\,8966\,98$. We wish to find integers m, n satisfying

$$365{\cdot}2421\,8966\,98 \times m \approx n$$

or

$$(365 + 0{\cdot}2421\,8966\,98)\ m \approx n$$

and therefore

$$0{\cdot}2421\,8966\,98\ m \approx n - 365\ m$$

which we write as

$$0{\cdot}2421\,8966\,98\ m \approx n' \tag{15}$$

with

$$n' = n - 365\ m$$

giving

$$n = n' + 365\ m \tag{16}$$

So we see from (15) we require integers m, n' such that

$$\frac{n'}{m} \approx 0{\cdot}2421\,8966\,98 \tag{17}$$

To find approximations n'/m we represent the decimal as a continued fraction. We obtain

$$0{\cdot}2421\,8966\,98 = [\,0;\ 4,\ 7,\ 1,\ 3,\ 27,\ 1,\ \ldots]$$

The partial quotients are therefore

$$a_0 = 0, \quad a_1 = 4, \quad a_2 = 7, \quad a_3 = 1, \quad a_4 = 3, \quad a_5 = 27, \quad a_6 = 1, \quad \ldots$$

and we obtain the principal convergents

$$\frac{0}{1},\quad \frac{1}{4},\quad \frac{7}{29},\quad \frac{8}{33},\quad \frac{31}{128},\quad \frac{845}{3489},\quad \frac{876}{3617},\quad \ldots \tag{18}$$

If we set these to A_i/B_i, $i = 0,\ 1,\ 2,\ \ldots$ respectively, the intermediate convergents can be found using (11). We find the intermediate convergents for $k = 0, 1, \ldots, 4$ to be

$$\left.\begin{array}{ll}
k=0 & \frac{1}{5},\ \frac{2}{9},\ \frac{3}{13},\ \frac{4}{17},\ \frac{5}{21},\ \frac{6}{25} \\
k=1 & \text{none} \\
k=2 & \frac{15}{62},\ \frac{23}{95} \\
k=3 & \frac{39}{161},\ \frac{70}{289},\ \frac{101}{417},\ \frac{132}{545},\ \frac{163}{673},\ \frac{194}{801},\ \frac{225}{929}, \\
 & \frac{256}{1057},\ \frac{287}{1185},\ \frac{318}{1313},\ \frac{349}{1441},\ \frac{380}{1569},\ \frac{411}{1697},\ \frac{442}{1825}, \\
 & \frac{473}{1953},\ \frac{504}{2081},\ \frac{535}{2209},\ \frac{566}{2337},\ \frac{597}{2465},\ \frac{628}{2593},\ \frac{659}{2721}, \\
 & \frac{690}{2849},\ \frac{721}{2977},\ \frac{752}{3105},\ \frac{783}{3233},\ \frac{814}{3361} \\
k=4 & \text{none}
\end{array}\right\} \tag{19}$$

The principal convergent 876/3617 agrees to seven decimal places with the original decimal value in (17). From (14) we see the tropical year is a slowly varying quantity with time T, the T term having a coefficient of the order of 10^{-6}, and so there is little point in extending the sequence of principal convergents and hence intermediate convergents.

Using the value for the mean length of the solar year 365·2422, as in article on the calendar in *Encyclopaedia Britannica* for 1911, we can express 0·2422 as a continued fraction. We find

$$0{\cdot}2422 = [0;\ 4,\ 7,\ 1,\ 3,\ 4,\ 1,\ \ldots]$$

giving principal convergents

$$\frac{0}{1},\quad \frac{1}{4},\quad \frac{7}{29},\quad \frac{8}{33},\quad \frac{31}{128},\quad \frac{132}{545},\quad \frac{163}{673},\quad \ldots \tag{20}$$

and intermediate convergents using (11) with $k = 0, 1, \ldots, 4$

$$\left.\begin{array}{ll}
k=0 & \frac{1}{5},\ \frac{2}{9},\ \frac{3}{13},\ \frac{4}{17},\ \frac{5}{21},\ \frac{6}{25} \\[2ex]
k=1 & \text{none} \\[2ex]
k=2 & \frac{15}{62},\ \frac{23}{95} \\[2ex]
k=3 & \frac{39}{161},\ \frac{70}{289},\ \frac{101}{417} \\[2ex]
k=4 & \text{none}
\end{array}\right\} \tag{21}$$

Note interestingly in the continued fraction for 0·2422, 132/545 and 163/673 are principal convergents but in the continued fraction for 0·2421 8966 98 these are intermediate convergents.

From the continued fraction for 0·2421 8966 98, which we denote as c, we consider from equation (17) some examples of n' and m.

Example 1

Consider the principal convergent $n' = 1$, $m = 4 \Rightarrow n = 1 + 365 \times 4$ using (16). So in every 4 years add 1 intercalary day or have 1 leap year. This was a rule in the Julian calendar. We note that the annual error would be $c - 1/4 \approx -0{\cdot}781 \times 10^{-2}$ which amounts to a gain of about 8 days in every thousand years.

Example 2

Consider the intermediate convergent $n' = 6$, $m = 25$. This is the same as taking $n' = 24$, $m = 100$ so $n = 24 + 365 \times 100$. So a better rule than that used in the Julian calendar would be to have in every 100 years to add 24 intercalary days or have 24 leap years. We could make each year divisible by 100 a normal year, whereas every other year divided by 4 a leap year. The annual error would be $c - 6/25 \approx 0{\cdot}219 \times 10^{-2}$ which amounts to the loss of about 2 days in every thousand years.

Example 3

Consider the values $n' = 97$, $m = 400$. The value 97/400 is not a principal nor an intermediate convergent. However, it is close to the intermediate convergent with $n' = 194$, $m = 801$. With $n' = 97$, $m = 400$ we have $n = 97 + 365 \times 400$. In every 400 years add 97 intercalary days, or have 97 leap years. This can be achieved by taking a leap year every 4 years *except* if year is divisible by 100 and *not* by 400. This rule is used in our current calendar, the Gregorian calendar. The annual error would be $c - 97/400 \approx -0{\cdot}310 \times 10^{-3}$ which amounts to a gain of about 3 days in every ten thousand years, a considerable improvement to the Julian calendar. We note if we took the intermediate convergent value 194/801 then $c - 194/801 \approx -0{\cdot}758 \times 10^{-5}$ which now gives a gain of about 8 days every million years. Creating a rule in the calendar to implement this would be much more complicated and differ markedly from previous calendar rules.

In the Appendix we give some details of the work by Rockett and Szüsz (1992) pages 60 to 62 who show in their book how the approximation 97/400 can be determined and in general how an approximation a/b to c can be found when b is not a denominator of a convergent of c.

Example 4

Consider the principal convergent $n' = 31$, $m = 128 \Rightarrow n = 31 + 365 \times 128$. In every 128 years add 31 intercalary days, or have 31 leap years. This could be implemented by making every fourth year a leap year but with the thirty second consecutive set of 4 years a normal year. The annual error would be $c - 31/128 \approx 0{\cdot}217 \times 10^{-5}$ giving a loss of about 2 days every million years. Although this is a great improvement to the Gregorian calendar it would be easier to implement if our number system was using base 2 arithmetic and not base 10.

3.2 **Lunar calendars**

The *synodic month*, the mean interval between conjunctions of the Moon and Sun, corresponds to the cycle of lunar phases. The following expression for the synodic month in days given in the *Explantory Supplement to the Astronomical Almanac*, 2013 p.587 and based on the lunar theory of Chapront-Touzé and Chapront (1988) is:

$$29{\cdot}5305\,8885\,31 + 0{\cdot}0000\,0021\,621\,T - 3{\cdot}64 \times 10^{-10}\,T^2 \tag{22}$$

with $T = (JD - 245\,1545{\cdot}0)/36525$ and JD is the Julian date.

Consider a year to consist of 12 lunar months, as is done in the Islamic calendar. Taking the synodic month from (22) with $JD = 245\,1545{\cdot}0$ then

$$\begin{aligned} 1 \text{ year} &= 12 \times 29{\cdot}5305\,8885\,31 \text{ days} \\ &= 354{\cdot}3670\,6623\,72 \text{ days} \end{aligned} \tag{23}$$

To find how many years equal approximately a multiple of tropical years we must find integers m, n such that

$$354{\cdot}3670\ 6623\,72 \times m \approx 365{\cdot}2421\ 8966\ 98 \times n \tag{24}$$

that is

$$\frac{354{\cdot}3670\ 6623\,72}{365{\cdot}2421\ 8966\ 98} \approx \frac{n}{m}$$

or

$$0{\cdot}9702\ 2489\ 81 \approx \frac{n}{m} \tag{25}$$

To find approximations n/m we represent the decimal as a continued fraction. We obtain

$$0{\cdot}9702\ 2489\ 81 = [0;\ 1,\ 32,\ 1,\ 1,\ 2,\ \ldots]$$

The partial quotients are therefore

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 32, \quad a_3 = 1, \quad a_4 = 1, \quad a_5 = 2, \quad \ldots$$

and we obtain the principal convergents

$$\frac{0}{1},\quad \frac{1}{1},\quad \frac{32}{33},\quad \frac{33}{34},\quad \frac{65}{67},\quad \frac{163}{168},\quad \ldots \tag{26}$$

and intermediate convergents using (11) with $k = 0,\ 1,\ 2,\ 3$

$$\left.\begin{array}{lll} k = 0 & \dfrac{p}{p+1}, & p = 1, \ldots, 31 \\[2ex] k = 1,\ 2 & \text{none} & \\[2ex] k = 3 & \dfrac{98}{101} & \end{array}\right\} \tag{27}$$

The convergent 163/168 agrees with the original decimal in (25) to 4 decimal places.

Note taking the principal convergent 32/33 we see 33 years in the Islamic calendar is 6·3631 1639 40 days greater than 32 tropical years and if the principal convergent 163/168 is taken we see 168 years in the Islamic calendar is 0·8097 8832 78 days less than 163 tropical years. Thus in 33 years each month goes the round of the seasons and over the longer interval of 168 years follows the seasons closer still.

For astronomical purposes the following rule is applied. In a cycle of 30 years in the Islamic calendar each year has 12 months alternating in length of 30 days and 29 days except for the twelfth month which has 29 days 19 times and 30 days 11 times. There are thus 11 leap years in a cycle of 30 years.

We can see how this cycle of years originates by considering the equation

$$354\,m + 355\,n \approx (m+n)\,354{\cdot}3670\ 6623\ 72 \tag{28}$$

where m and n are integers. From (28) we find

$$0{\cdot}5799\ 4415\ 65 \approx \frac{n}{m} \tag{29}$$

Approximations n/m can be found if we represent the decimal as a continued fraction. We find

$$0{\cdot}5799\,4415\,65 = [0;\ 1,\ 1,\ 2,\ 1,\ 1,\ 1,\ 2,\ 6,\ \ldots]$$

The partial quotients are therefore

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 1, \quad a_3 = 2, \quad a_4 = 1, \quad a_5 = 1, \quad a_6 = 1, \quad a_7 = 2, \quad a_8 = 6, \quad \ldots$$

and we obtain the principal convergents

$$\frac{0}{1},\quad \frac{1}{1},\quad \frac{1}{2},\quad \frac{3}{5},\quad \frac{4}{7},\quad \frac{7}{12},\quad \frac{11}{19},\quad \frac{29}{50},\quad \frac{185}{319},\quad \ldots \tag{30}$$

and intermediate convergents using (11) with $k = 0, 1, \ldots, 6$

$$\left.\begin{array}{ll} k=0 & \text{none} \\ k=1 & \dfrac{2}{3} \\ k=2,\ 3,\ 4 & \text{none} \\ k=5 & \dfrac{18}{31} \\ k=6 & \dfrac{40}{69},\quad \dfrac{69}{119},\quad \dfrac{98}{169},\quad \dfrac{127}{219},\quad \dfrac{156}{269} \end{array}\right\} \tag{31}$$

The sequence of principal convergents slowly approaches the decimal value in (29). This can be anticipated by the number of partial quotients with the value of 1. The eighth principal convergent, 185/319, agrees with this decimal to only 4 decimal places.

We note if we consider in particular the principal convergent 11/19 then from (28) we can have 19 years of 354 days and 11 years of 355 days together which closely equal a cycle of $11 + 19 = 30$ years in the Islamic calendar. In this cycle there will be 360 lunations in 10,631 days. The true length of 360 lunations is $10,631{\cdot}0119\,8711\,60$ days or 1 day error in 2,500 years.

3.3 **Luni-solar calendars**

Luni-solar calendars are based on the lunar month and the tropical year. We ask what multiples of tropical years are approximately equal to a multiple of synodic months? So from (14) and (22) for $JD = 245\ 1545{\cdot}0$ we wish to find integers m, n satisfying

$$365{\cdot}2421\ 8966\ 98 \times m \approx 29{\cdot}5305\ 8885\ 31 \times n \tag{32}$$

From (32) we have

$$\frac{29{\cdot}5305\ 8885\ 31}{365{\cdot}2421\ 8966\ 98} \approx \frac{m}{n}$$

or

$$0{\cdot}0808\ 5207\ 48 \approx \frac{m}{n} \tag{33}$$

We can express the decimal value in (33) as a continued fraction. We find

$$0{\cdot}0808\,5207\,48 = [0;\ 12,\ 2,\ 1,\ 2,\ 1,\ 1,\ 17,\ \ldots]$$

The partial quotients are therefore

$$a_0 = 0, \quad a_1 = 12, \quad a_2 = 2, \quad a_3 = 1, \quad a_4 = 2, \quad a_5 = 1, \quad a_6 = 1, \quad a_7 = 17, \quad \ldots$$

from which we find the principal convergents

$$\frac{0}{1},\quad \frac{1}{12},\quad \frac{2}{25},\quad \frac{3}{37},\quad \frac{8}{99},\quad \frac{11}{136},\quad \frac{19}{235},\quad \frac{334}{4131},\quad \ldots \tag{34}$$

and using (11) we find the intermediate convergents for $k = 0, 1, \ldots, 5$

$$\left.\begin{array}{ll}
k = 0 & \dfrac{1}{13} \\[2ex]
k = 1 & \text{none} \\[1ex]
k = 2 & \dfrac{5}{62} \\[2ex]
k = 3, 4 & \text{none} \\[1ex]
k = 5 & \dfrac{30}{371}, \; \dfrac{49}{606}, \; \dfrac{68}{841}, \; \dfrac{87}{1076}, \; \dfrac{106}{1311}, \; \dfrac{125}{1546}, \; \dfrac{144}{1781}, \; \dfrac{163}{2016}, \\[2ex]
 & \dfrac{182}{2251}, \; \dfrac{201}{2486}, \; \dfrac{220}{2721}, \; \dfrac{239}{2956}, \; \dfrac{258}{3191}, \; \dfrac{277}{3426}, \; \dfrac{296}{3661}, \; \dfrac{315}{3896}
\end{array}\right\} \qquad (35)$$

The principal convergent 334/4131, as a decimal, agrees to 6 significant figures with the decimal in (33).

All Greek calendars were lunar until the Roman period. From the sixth century BC onwards Greek astronomers devised cycles in an attempt to maintain both the mean month and mean year in the cycle with their astronomical values. In the cycles there were rules on the number of days in the months and the number of months in the years. The following brief description of cycles used in Greek calendars and some of the cycles in the ecclesiastical calendar are obtained from an article by Fotheringham (1935). This article was a revision of an earlier article on the calendar, Fotheringham (1931). As well as revisions, a new paragraph was added on *Subdivisions of the day*. The information used here on the Greek cycles was unchanged in this revision. We show how these are linked with some of the convergents found above.

In Greek calendars a number of cycles were used relating a number of years to a number of lunations and these in turn were set to a number of days. Cleostratus of Tenedos in the sixth century BC introduced the octaeteris or 8-year cycle (uses principal convergent 8/99). It made 8 years = 99 lunations which were set to 2922 days. A number of modifications were made to calendars based on this cycle. The first was to set 16 years (2 octaeteris cycles) = 198 lunations = 5847 days ($2 \times 2922 + 3$) and another was to put 160 years (20 octaeteris cycles) = 1979 lunations ($20 \times 99 - 1$) = 58 440 days ($10 \times (2 \times 2922 + 3) - 30) = 20 \times 2922$.

A great advance was made moving to the approximation corresponding to the sixth principal convergent, which is a 19-year cycle (uses principal convergent 19/235) so 19 years = 235 lunations and were made equal to 6940 days. It was introduced by Meton of Athens and began on June 27 432 BC. Callippus attempted to improve on the calendar based on the Metonic cycle by combining four 19-year periods to form a period of 76 years. He made 76 years = 940 lunations which he set equal to 27 759 days ($4 \times 6940 - 1$). The first Callipic cycle was made to begin in 330 BC when the summer solstice and New Moon coincided. The last of the Greek astronomical cycles was that devised by Hipparchus, but appears never to have been used. For this cycle he set 304 years (4 Callippic cycles) = 3760 lunations and set it equal to 111 035 days ($4 \times 27\,759 - 1$).

Data on each cycle, as discussed above, is gathered in Table 1, together with for each cycle the mean length of a solar year and the mean length of a lunar month and the differences of these quantities with the true mean tropical year and lunar month respectively.

The octaeteris, 160 year cycle and the Callippic cycle all have a mean year of 365·25 days, as in the Julian calendar. The cycle of Hipparchus is seen to represent closest the true length of a tropical year and a lunar month. In a cycle, the mean year is too long by 6 minutes 30·6 seconds and the mean lunar month too short by 0·3 seconds!

In the computation of the date of Easter various rules have been employed using cycles of 19 years (Metonic), 84years (Roman) and 532 years. In the Roman cycle, 84 years = 1039 lunations and was set to equal 30 681 days. The fraction 84/1039 is neither a principal nor an intermediate convergent. It is close to the intermediate convergent 87/1076.

It is interesting to note that the Roman and the Greek cycles above other than the octaeteris and Metonic cycles, which originate from the principal convergents, can be obtained by applying the method described in the Appendix. In applying it we take c to be the decimal in equation (33) and use its principal convergents in equation (34).

Table 1: Greek Cycles

Cycle	Originator	No. of solar years	No. of lunar months	Total No. of days	Mean year	Mean lunar month	Difference: true mean tropical year − mean year	lunar month − mean lunar month
					days	days	days	days
Octaeteris	Cleostratus	8	99	2 922	365·25	29·515 152	$-0{\cdot}78 \times 10^{-2}$	$+0{\cdot}15 \times 10^{-1}$
	Geminus†	16	198	5 847	365·4375	29·530 303	$-0{\cdot}20$	$+0{\cdot}29 \times 10^{-3}$
	Geminus†	160	1979	58 440	365·25	29·530 066	$-0{\cdot}78 \times 10^{-2}$	$+0{\cdot}52 \times 10^{-3}$
Metonic	Meton	19	235	6 940	365·263 158	29·531 915	$-0{\cdot}21 \times 10^{-1}$	$-0{\cdot}13 \times 10^{-2}$
Callippic	Callippus	76	940	27 759	365·25	29·530 851	$-0{\cdot}78 \times 10^{-2}$	$-0{\cdot}26 \times 10^{-3}$
	Hipparchus	304	3760	111 035	365·246 711	29·530 585	$-0{\cdot}45 \times 10^{-2}$	$+0{\cdot}37 \times 10^{-5}$

† Contained in the records of Geminus.

Thus, for example, to derive the Roman cycle we seek an approximation $a/1039$ (thus $b = 1039$) and applying the three steps find

$$\begin{aligned} b &= 4\,B_6 + 0\,B_5 + 1\,B_4 + 0\,B_3 + 0\,B_2 + 0\,B_1 + 0\,B_0 \\ A &= 4\,A_6 + 1\,A_4 \\ &= 84 \\ b\,c - A &= 0{\cdot}0053\;0571\;72 \end{aligned}$$

giving $a = A = 84$ and so to the approximation $84/1039$.

The cycle of $532\,(= 19 \times 28)$ years was based on the Metonic cycle by Victorius in AD 457. The Metonic cycle is the only one of these cycles still used in the computation of the date of Easter.

It is also used today in the modern Jewish calendar. A description of this calendar is given by Richards (2013) pp 601-6.

4. **Conclusions**

This technical note uses continued fractions to identify astronomical periods used in some of the leading calendars. The first part of the note describes basic theory on continued fractions including terminology and nomenclature used and then concentrating on the computation of principal and intermediate convergents. Some discussion is then given to what we mean by a best approximation to a positive number. Examples of continued fractions to illustrate these techniques are presented. In the discussion of astronomical periods in solar, lunar and luni-solar calendars we do not give details on the history of any one particular calendar. We refer the reader for this information to other sources such as Richards (2013) and Fotheringham (1935).

An interesting question is how did the early calendar-makers acquire the knowledge of the basic periods of the solar year, lunar month and the ratio of these two quantities? In the work of Zeeman(1986) on explaining the astronomical significance of the gear trains in the Antikythera Mechanism he put forward an algorithm on how the Metonic cycle may have been calculated. Observe the number of new moons in a year and record this number. A *new year* may be defined as the first day of the year a particular chosen bright star is visible at sunset. Record these numbers over a period of time. Zeeman did this for 65 years. Each number in the sequence will be a 12 or 13. Next count the interval between successive 13's in this sequence. You obtain another sequence this time each number being a 2 or 3. In this new sequence count the intervals between successive 3's and a third sequence is obtained with each number being a 1 or 2. Continuing in this manner the next sequence will contain the numbers 2 and 3 and the one following also 2 and 3. Each sequence will be shorter than the previous one. Now take the smallest number in each sequence, that is, 12, 2, 1, 2 and 2 and form the continued fraction $[\,12; 2, 1, 2, 2\,]$. We find the Metonic ratio 235/19. There is an alternative computation of this cycle, put forward by Goldstein (1966), which is derived from basic assumptions on the length of the year and the synodic month.

From observing new moons in a new year over a period of time, Zeeman effectively finds a rational approximation to the sidereal year divided by the synodic month. Interestingly, as Zeeman remarked in his paper, although this ratio is found to a great accuracy, the sidereal year and synodic month do not need to be known accurately. In section 3.3 the Metonic ratio was a convergent in the continued fraction for the tropical year divided by the synodic month. Now we note from *The Astrononmical Almanac 2000* the sidereal year for J2000·0 = JD 245 1545·0 is 365·2563 63 days, approximately 20 minutes longer than the tropical year, see section 3.1. In fact the sidereal year varies slowly in time, see *Explanatory Supplement* (1961) p.99. The convergents in the continued fraction for the sidereal year divided by the synodic month found by Zeeman are 12/1, 25/2, 37/3, 99/8, and 235/19 and from section 3.3 equation (34) the convergents for the tropical year divided by the synodic month are 12/1, 25/2, 37/3, 99/8, 136/11, 235/19, ... Thus the two continued fractions have the same principal convergents up to the Metonic cycle except in Zeeman's continued fraction the approximation 136/11 is now an intermediate convergent and not a principal convergent.

In addition to the Metonic ratio the octaeteris ratio 99/8 is also a principal convergent in Zeeman's continued fraction. As discussed in section 3.3 we know the remaining Greek cycles, shown in Table 1, have numerators and denominators a linear combination of the numerators and denominators of the Metonic and octaeteris ratios, in fact except for the ratio 1979/160 they are just multiples of one or other of these ratios. So from Zeeman's continued fraction we can determine all Greek cycles in Table 1.

A theorem to prove why the algorithm described above to calculate the Metonic ratio works was given in Zeeman (1986). The converse of this theorem has been known for a long time, see Christoffel (1875) and Smith (1876). As an illustration of this we look at the continued fraction for the tropical year at time $JD = 245\,1545{\cdot}0$. From section 3.1 we see its value is 365·2421 8966 98 which can be written as the continued fraction $[\,365;\, 4,\, 7,\, 1,\, 3,\, 27,\, 1, \ldots]$. Now consider counting the number of whole days in a tropical year starting from a point at which the start of the first day and start of the year are coincident and with the fraction of the day left over in the year carried on to the next year. We will obtain a sequence of years, the length of the year being either 365 or 366 days. So the first ten years in this sequence will be

$$365, \quad 365, \quad 365, \quad 365, \quad 366, \quad 365, \quad 365, \quad 365, \quad 366, \quad 365, \ldots \tag{36}$$

Now count the steps between each 366 and the next. If our original sequence in (36) is taken far enough we obtain the sequence

$$4, \quad 4, \quad 4, \quad 4, \quad 4, \quad 4, \quad 5, \quad 4, \quad 4, \quad 4, \quad 4, \quad 4, \quad 4, \quad 4, \quad 5, \quad 4, \quad \ldots$$

This sequence is made up of the numbers 4 and 5. Now count the number of steps between each 5 and the next. This sequence is found to contain the numbers 7 and 8. A small computer program was written to calculate these sequences. Continuing in this manner the next sequence we find contains the numbers 1 and 2 and the one following 3 and 4. To establish successive sequences requires the span of years to be increased. For example, the first 4 appears in the sequence containing the numbers 3 and 4 requires an original sequence made up of the numbers 365 and 366 to cover 3,684 years. The smaller numbers in each sequence are the partial quotients in the continued fraction for the tropical year.

It should be remarked that the cycles and rules used in calendars established here could equally have been determined by writing a small computer program to find the best rational approximations to a given irrational number. In this program all rational numbers p/q, where p and q can take on all positive integers less than a designated integer are each compared with the irrational number. However, a greater understanding of the structure of these approximations can best be found by using the theory of continued fractions. It is the intention of this technical note to show the ease in computing such approximations.

5. Acknowledgements

The section on explaining the Greek cycles follows the approach in an unpublished note of a talk on Calendars given to the author by the late Dr P.J. Message. I would like to thank Catherine Hohenkerk for her expertise in TEX which produced this final form of the Technical Note.

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Appendix A — The Method of Rockett and Szüsz

The method of Rockett and Szüsz to determine a rational approximation a/b to c, where c is the decimal part of the mean tropical year, and b is not a denominator of a convergent of c.

This method was devised by Rockett and Szüsz (1992) and can be found on pages 60 to 63 of their book. In deriving this method some mathematics is required beyond the scope of this technical note. We give here only the steps to apply this method and then use it to discuss various approximations to c.

Let c be the decimal part of the mean tropical year (see section 3.1). Suppose b is a positive integer not equal to a denominator of any principal convergent of c. We seek an integer a such that $|bc - a|$ is as small as possible. The method of Rockett and Szüsz can be broken down into three steps.

The first step uses *Ostrowski's algorithm* (see Ostrowski (1921)). Suppose B_0, B_1, B_2, ... are the denominators of the principal convergents of c. Thus we have $1 = B_0 \leq B_1 < B_2 < B_3 < \ldots$ and there will be an index N so that $B_N \leq b < B_{N+1}$. Now we can write $b = d_{N+1}B_N + R$, where d_{N+1} is the greatest integer not exceeding b/B_N, and the remainder R is $0 \leq R < B_N$. If $R > 0$ we can repeat this process expressing the remainder R as a multiple of another B_k for some $k < N$ and another remainder and continue this process until we eventually get the remainder to be zero. This procedure is called Ostrowski's algorithm and allows us to express b as

$$b = \sum_{k=0}^{N} d_{k+1}\, B_k \tag{A.1}$$

The d_{k+1}'s are unique and this expression for b is termed the *Ostrowski representation* of b.

In the second step, using the numerators A_0, A_1, A_2, A_3, ... of the principal convergents of c, we define the number

$$A = A(b) = \sum_{k=0}^{N} d_{k+1}\, A_k \tag{A.2}$$

From (A.1) and (A.2) we find

$$|bc - A| < 1 \tag{A.3}$$

This follows from a proof of a theorem given in Rockett and Szüsz p.26.

The third step follows from (A.3). We define the integer a as

$$a = a(b) = \begin{cases} A & \text{if } |bc - A| \leq 1/2 \\ A+1 & \text{if } bc - A > 1/2 \\ A-1 & \text{otherwise} \end{cases} \tag{A.4}$$

Some examples will help demonstrate how to apply this method. The principal convergents A_k/B_k, $k = 0, 1, \ldots, 6$ for $c = 0{\cdot}2421\ 8966\ 98$ are given in equation (18) in section 3.1.

Example A.1

Consider case with $b = 400$. This was an example used by Rockett and Szüsz. From step 1 we find

$$b = 3\,B_4 + 0\,B_3 + 0\,B_2 + 4\,B_1 + 0\,B_0$$

and from step 2

$$\begin{aligned} A &= 3\,A_4 + 4\,A_1 \\ &= 97 \end{aligned}$$

From the third step we find

$$bc - A = -0{\cdot}1241\ 3208\ 00$$

and so from (A.4) we see $a = A$ and obtain therefore the approximation 97/400 to c.

Example A.2

Consider case with $b = 700$. From step 1 we find

$$b = 5\,B_4 + 1\,B_3 + 0\,B_2 + 6\,B_1 + 3\,B_0$$

and from step 2

$$A = 5\,A_4 + 1\,A_3 + 6\,A_1 + 3\,A_0$$
$$= 169$$

The third step we find
$$b\,c - A = 0{\cdot}5327\ 6886\ 00$$

and so from (A.4) we see $a = A + 1$ and obtain therefore the approximation 170/700 to c.

In Table A.1 approximations a/b to c are given for values of b at 100 year intervals from 100 years to 1000 years. The quantity $c - a/b$ is also listed for each approximation to show the accuracy and also to enable comparison of them.

Table A.1: Approximations a/b to the decimal fraction of the mean tropical year c and measure of the approximation $c - a/b$

a/b	$c - a/b$
$\frac{24}{100}$	$+0{\cdot}0021\ 8966\ 98$
$\frac{48}{200}$	$+0{\cdot}0021\ 8966\ 98$
$\frac{73}{300}$	$-0{\cdot}0011\ 4366\ 35$
$\frac{97}{400}$	$-0{\cdot}0003\ 1033\ 02$
$\frac{121}{500}$	$+0{\cdot}0001\ 8966\ 98$
$\frac{145}{600}$	$+0{\cdot}0005\ 2300\ 31$
$\frac{170}{700}$	$-0{\cdot}0006\ 6747\ 31$
$\frac{194}{800}$	$-0{\cdot}0003\ 1033\ 02$
$\frac{218}{900}$	$-0{\cdot}0000\ 3255\ 24$
$\frac{242}{1000}$	$+0{\cdot}0001\ 8966\ 98$

The approximation 97/400 which leads to the rule in the Gregorian calendar has already been discussed in Example 3 in section 3.1. The best fit of the approximations to c given in Table A.1 is 218/900 and is followed by 121/500. The former approximation indicates a calendar based on this approximation to have two extra leap years and these could be at the 400th year and 900th year in the 900 year cycle. Thus implementing this rule we would alternate between 400 and 500 years when the extra leap years are to be applied. We would now have a gain of about 3 days every 100 000 years. The approximation 121/500, which was identified by Rockett and Szüsz (1992), pp 62-63, would be implemented with the extra leap year at the end of every fifth century not after every fourth as in the Gregorian calendar. There would now be a loss of about 2 days every 10 000 years.

In the Gregorian calendar, as shown in Example 3 of section 3.1, we expect an error amounting to a gain of 3 days in every ten thousand years. If we continued the uniformity of the inclusion of leap years to depend on the number 4 then it has been suggested (see Enyclopaedia Britannica, 1911) to make the years 4000, 8000, 12 000, . . . common years. We show why this is not appropriate.

From equation (14) in section 3.1 we see the length of the mean tropical year decreases slowly with time. Note the principal convergent $A_7/B_7 = 6101/25191$ to c following the previous convergent 876/3617, see equation (18), enables us to seek a best approximation $a/4000$ to c using the Rockett and Szüsz method. Following the steps of this method we find

$$4000 = 1\,B_6 + 0\,B_5 + 2\,B_4 + 3\,B_3 + 0\,B_2 + 7\,B_1 + 0\,B_0$$

$$
\begin{aligned}
A &= 1\,A_6 + 2\,A_4 + 3\,A_3 + 7\,A_1 \\
&= 969
\end{aligned}
$$

and finally as

$$4000\,c - 969 = -0{\cdot}2413\;2080\;00$$

we see

$$a = A = 969$$

Thus we find $c - (969/4000) = -0{\cdot}0000\;6033\;02$, giving a gain of 6 days every 100 000 years.

Using equation (14) in section 3.1 and choosing a value of $T = 40$, we can compute a mean tropical year forty Julian centuries in the future. We obtain a value for c then in this case of 0·2419 5925 58 and following an analogous argument to above seek a best approximation to this of the form $a/4000$. From the continued fraction for 0·2419 5925 58 we compute the principal convergents and then apply the method of Rockett and Szüsz and obtain an approximation 968/4000. Thus we see the best approximation of the form $a/4000$ to the mean tropical year in 40 Julian centuries time is not the same as that to the current value of the mean tropical year. The approximation 97/400 does though remain valid for both values of the mean tropical year. Any rules to adjust the Gregorian calendar must take into consideration the decrease in the mean tropical year over time.