Computation of the Quantities Describing the Lunar Librations in The Astronomical Almanac

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1. Introduction

Lunar physical ephemerides, tabulated daily including libration, can be found in Section D of *The Astronomical Almanac* (AsA). This technical note describes the method used for calculating the lunar librations and other quantities on the odd pages D7–D21 of the AsA with effect from the 2011 edition. Prior to the 1985 edition, formulae and constants for the physical ephemeris of the Moon were due to Hayn (1907). Beginning with the 1985 edition, librations have been generated using the analytical theory of Eckhardt (1981, 1982).

Since the inclusion of a rotational ephemeris for the Moon given in the form of three Euler angles in the JPL lunar ephemerides starting with DE403/LE403, Standish et al. (1995), it has been a long-standing goal to implement these ephemerides in the calculation of lunar librations and improve the quality of the tabulated physical ephemerides of the Moon in Section D. Work started by Hilton (2004) on interpreting these ephemerides has paved the way to the work presented here which describes the calculation of librations and related quantities as they have been implemented starting with the 2011 edition of the AsA. This technical note has been written with the practitioner of astronomical calculations in mind and may form the basis of explanatory material on lunar librations in a future edition of *The Explanatory Supplement to the Astronomical Almanac* (ES).

In this technical note the basic formulae are derived for the lunar librations and the position angle of the axis of rotation. The lunar rotation angles in the JPL ephemeris must be transformed by a series of rotations and then using simple vector algebra to angular quantities which can be substituted into the basic formulae. These will then give the true or total lunar librations. From these and the optical (geometric) librations, the physical librations can be computed. For the 2011 edition, the lunar rotation angles in DE403/LE403 are used due to the availability of certain ephemeris-specific transformations (Konopliv et al. (2001)).

An algorithm and numerical example, using the same computer routines used for the AsA, showing all the relevant stages in the computation of the librations and related quantities is given. Appendix C contains a short description of the calculation of the position angle of the Moon's bright limb and its phase or fraction illuminated – geometrical quantities tabulated on the odd pages D7–D21 of Section D.

In the ES (1961) an approximate method was given to calculate the lunar librations but with no details of its derivation. In Appendix B these formulae are derived based on the method in Encke (1843). This method was used primarily before the advent of high-speed electronic computers and is not used in the publications currently to calculate the lunar librations. It is included here for the interested reader.

2. Basic formulae for the lunar librations and the position angle of the axis

2.1 The lunar librations

The mean rotational state of the Moon is described by Cassini's empirical laws, which state that the descending node of the Moon's equator coincides with the ascending node of the Moon's orbit on the ecliptic; the Moon's equator maintains a constant inclination to the ecliptic; and the rotation rate is such that on average the same side is always facing the Earth. Thus the rotational rate must be equal to the rate of motion of the Moon's mean longitude. The actual rotation state has small periodic variations from this mean state caused by dynamical perturbations, and these cause the physical librations of the Moon's orientation. In addition there are the much larger optical librations in its orientation as seen from the Earth, which are due to variations in the rate of the Moon's orbital motion, and to the inclination of the Moon's equator to its orbital plane (see Appendix A for an estimation of the magnitude of these librations). For a complete definition of the rotational state of the Moon, a prime meridian must also be specified, and this was originally chosen to be the mean central meridian of the side facing the Earth. Its direction in space will thus differ by 180° from the mean longitude of the Moon. The situation is illustrated in Figure 1.

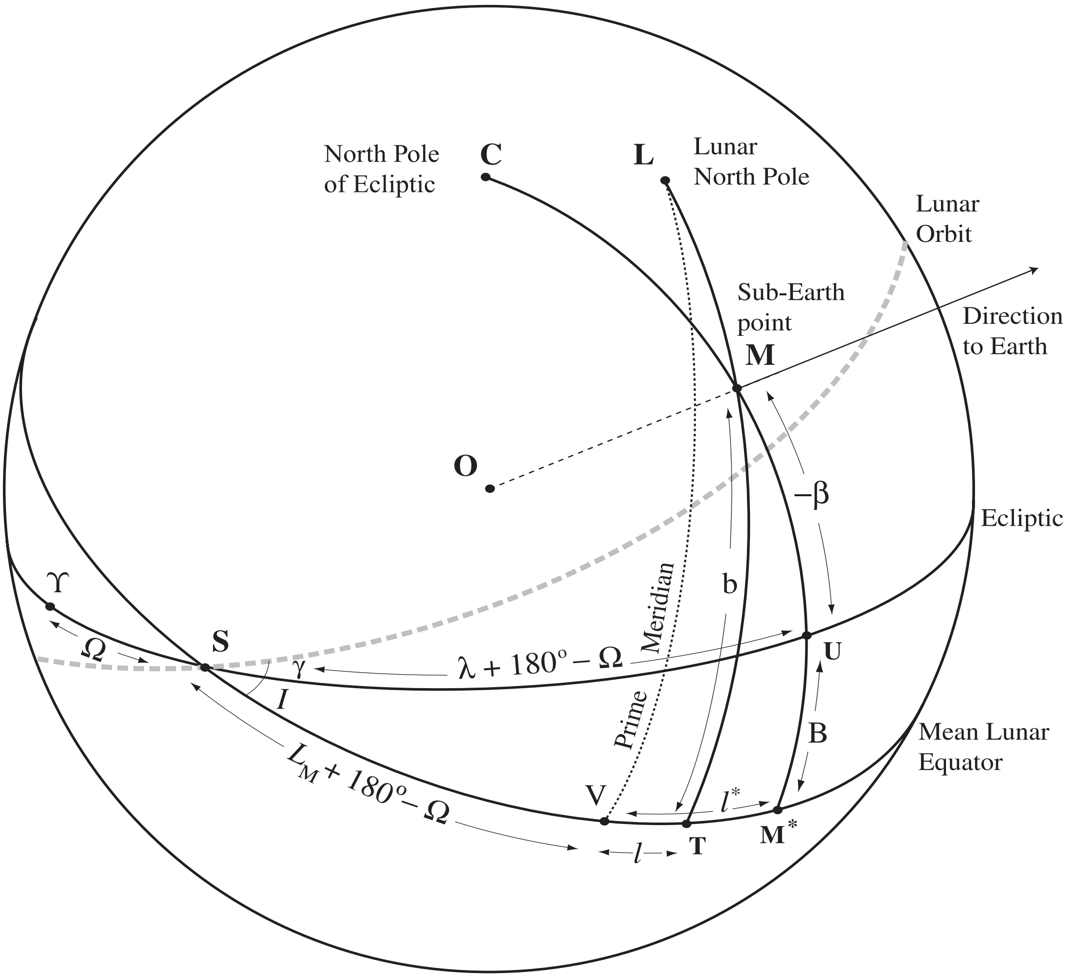

Figure 1: The selenocentric sphere: showing the lunar orbit and the relationships between the sub-Earth point M*, the mean lunar equator and the ecliptic.* S *is the descending node of the lunar equator on the ecliptic.*

The ecliptic longitude and latitude of the Moon are λ, β respectively and so the sub-Earth point M has longitude and latitude $\lambda + 180°$, $-\beta$. L_M is the mean longitude of the Moon and Ω is the ascending node. The inclination of the ecliptic to the mean lunar equator is I. The librations in longitude and latitude are denoted by l and b respectively.

Formulae for computing the optical librations can be derived by relating to each other two expressions for the vector from the centre of the Moon towards the sub-Earth point M; one of them referred to the ecliptic frame and the other to the lunar equatorial frame. Let O be the centre of the selenocentric sphere. In Figure 1 consider firstly point M referred to right-handed axes O_{xyz} in which O_x is in direction OS, O_y in the ecliptic and O_z in direction OC. Coordinates of M in this system are

$$\begin{pmatrix} \cos(-\beta)\cos(\lambda + 180^\circ - \Omega) \\ \cos(-\beta)\sin(\lambda + 180^\circ - \Omega) \\ \sin(-\beta) \end{pmatrix} \tag{1}$$

Consider next, point M referred to axes $O_{x'y'z'}$ in which $O_{x'}$ is in direction OS, $O_{y'}$ in the mean equator of the Moon and $O_{z'}$ in the direction OL. The coordinates of M referred to these axes are

$$\begin{pmatrix} \cos b\cos(L_M + 180^\circ - \Omega + l) \\ \cos b\sin(L_M + 180^\circ - \Omega + l) \\ \sin b \end{pmatrix} \tag{2}$$

The vectors in equations (1) and (2) are related through the rotation matrix

$$\mathbf{R}_1(-I) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos I & -\sin I \\ 0 & \sin I & \cos I \end{pmatrix} \tag{3}$$

We note here in general that rotations in a right-handed set of axes with origin O about the O_x,O_y and O_z axes through an arbitrary angle θ are obtained by the rotation matrices $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}_3$ respectively, where

$$\mathbf{R}_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{4}$$

$$\mathbf{R}_2(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \tag{5}$$

$$\mathbf{R}_3(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6}$$

From equations (1), (2) and (3) we have

$$\begin{pmatrix} \cos b\cos(L_M + 180^\circ - \Omega + l) \\ \cos b\sin(L_M + 180^\circ - \Omega + l) \\ \sin b \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos I & -\sin I \\ 0 & \sin I & \cos I \end{pmatrix} \begin{pmatrix} \cos(-\beta)\cos(\lambda + 180^\circ - \Omega) \\ \cos(-\beta)\sin(\lambda + 180^\circ - \Omega) \\ \sin(-\beta) \end{pmatrix}$$

which can be written as

$$\begin{pmatrix} -\cos b\cos(L_M + l - \Omega) \\ -\cos b\sin(L_M + l - \Omega) \\ \sin b \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos I & -\sin I \\ 0 & \sin I & \cos I \end{pmatrix} \begin{pmatrix} -\cos\beta\cos(\lambda - \Omega) \\ -\cos\beta\sin(\lambda - \Omega) \\ -\sin\beta \end{pmatrix} \tag{7}$$

In the lunar ephemeris λ is referred to the true equinox of date, but L_M and Ω are referred to the mean equinox. The quantity $\lambda - \Omega$ in equation (7) must be replaced by $\lambda - (\Omega + N)$, where N is the nutation in longitude. We can now write (7) as the three equations

$$\cos b\cos(l + L_M - \Omega) = \cos\beta\cos(\lambda - \Omega - N) \tag{8}$$

$$\cos b\sin(l + L_M - \Omega) = \cos I\cos\beta\sin(\lambda - \Omega - N) - \sin I\sin\beta \tag{9}$$

$$\sin b = -\sin I\cos\beta\sin(\lambda - \Omega - N) - \cos I\sin\beta. \tag{10}$$

Equations (8), (9) and (10) are rigorous formulae for the computation of the optical librations l and b, from the values of I, Ω and L_M, which describe the mean rotational state of the Moon, and from λ and β, the ecliptic coordinates of the Moon. However, as is explained in Section 3, if I, Ω and L_M are substituted by modified quantities that include the effects of the dynamical perturbations of the Moon's rotation, then these rigorous formulae will give the values of the total librations.

An approximate method to compute the librations l, b was given in the ES (1961) p.319 based on formulae introduced by Encke (1843). It is derived using the quantities B and l^* shown in Figure 1 and was used before the advent of fast electronic computers. A statement of the method and an outline of its derivation is given in Appendix B for the interested reader.

2.2 The position angle of the axis

The position angle of the axis of rotation is the angle that the lunar meridian through the apparent central point of the disk towards the north lunar pole forms with the celestial meridian through the central point, measured eastwards from the celestial north point of the disk.

In determining expressions for the position angle we use the elements of the mean lunar equator referred to the Earth equator.

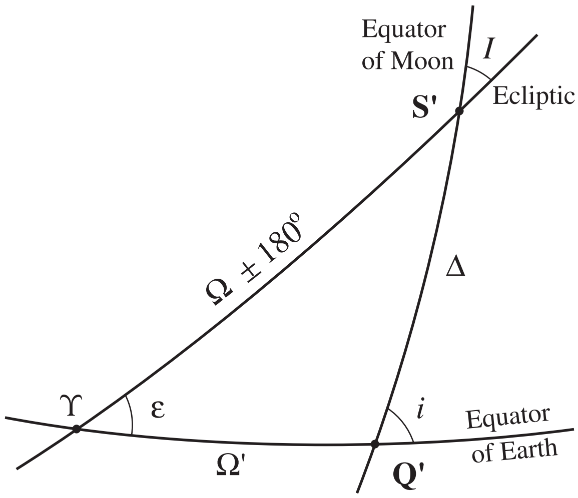

Figure 2: Elements for Moon's equator.

These are defined as:

i = the inclination of the mean equator of the Moon to the true equator of the Earth;

$\mathit{\Delta}$ = the arc of the mean equator of the Moon from its ascending node on the true equator of the Earth to its ascending node on the ecliptic of date;

$\mathit{\Omega}'$ = the arc of the true equator of the Earth from the true equinox of date to the ascending node of the mean equator of the Moon on the true equator of the Earth.

The ascending node of the mean lunar equator on the ecliptic is at the descending node of the mean lunar orbit so $\Upsilon S' = \mathit{\Omega} \pm 180°$. ε is the true obliquity and the node is referred to the true equinox by increasing $\mathit{\Omega}$ by the nutation in longitude N. From the spherical triangle $\Upsilon S'Q'$ in Figure 2 the elements can be found from the formulae

$$\sin\mathit{\Delta}\sin i = -\sin\varepsilon\sin(\mathit{\Omega}+N) \tag{11}$$
$$\cos\mathit{\Delta}\sin i = \sin I\cos\varepsilon - \cos I\sin\varepsilon\cos(\mathit{\Omega}+N) \tag{12}$$
$$\cos i = \cos I\cos\varepsilon + \sin I\sin\varepsilon\cos(\mathit{\Omega}+N) \tag{13}$$
$$\sin\mathit{\Omega}'\sin i = -\sin I\sin(\mathit{\Omega}+N) \tag{14}$$
$$\cos\mathit{\Omega}'\sin i = \cos I\sin\varepsilon - \sin I\cos\varepsilon\cos(\mathit{\Omega}+N). \tag{15}$$

In Figure 3 the position angle C' of the axis is shown on the selenocentric sphere. The geocentric right ascension and declination of the Moon are α, δ and so the right ascension and declination of the sub-Earth point are $\alpha + 180°$, $-\delta$. The descending node of the lunar equator on the ecliptic is denoted by point S. From the definition of $\mathit{\Delta}$ and $\mathit{\Omega}'$ the arcs $SQ = 360° - \mathit{\Delta}$ and $\Upsilon Q = \mathit{\Omega}' + 180°$. In the spherical triangle

NLM all angles and sides are known except angles $L\hat{N}M$ and $M\hat{L}N$. Noting that $XQ = 90°$ and $YQ = 90°$ these are found as follows:

$$
\begin{aligned}
L\hat{N}M &= 180° - YP \\
&= 180° - (\Upsilon P - \Upsilon Y) \\
&= 180° - \Upsilon P + (\Upsilon Q - YQ) \\
&= 180° - (\alpha + 180°) + (\Omega' + 180°) - 90° \\
&= \Omega' - \alpha + 90°
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
M\hat{L}N &= XT \\
&= XQ - TQ \\
&= XQ - (SQ - ST) \\
&= 90° - (360° - \Delta) + (L_M - \Omega + 180° + l) \\
&= \Delta + L_M + l - \Omega - 90°.
\end{aligned} \tag{17}
$$

The position angle (C') can be found from either of the two sets of formulae

$$\cos b \sin C' = -\sin i \cos(\Omega' - \alpha) \tag{18}$$
$$\cos b \cos C' = \cos\delta \cos i - \sin\delta \sin i \sin(\Omega' - \alpha) \tag{19}$$

or

$$\cos\delta \sin C' = \sin i \cos(L_M - \Omega + \Delta + l) \tag{20}$$
$$\cos\delta \cos C' = \cos i \cos b - \sin i \sin b \sin(L_M - \Omega + \Delta + l). \tag{21}$$

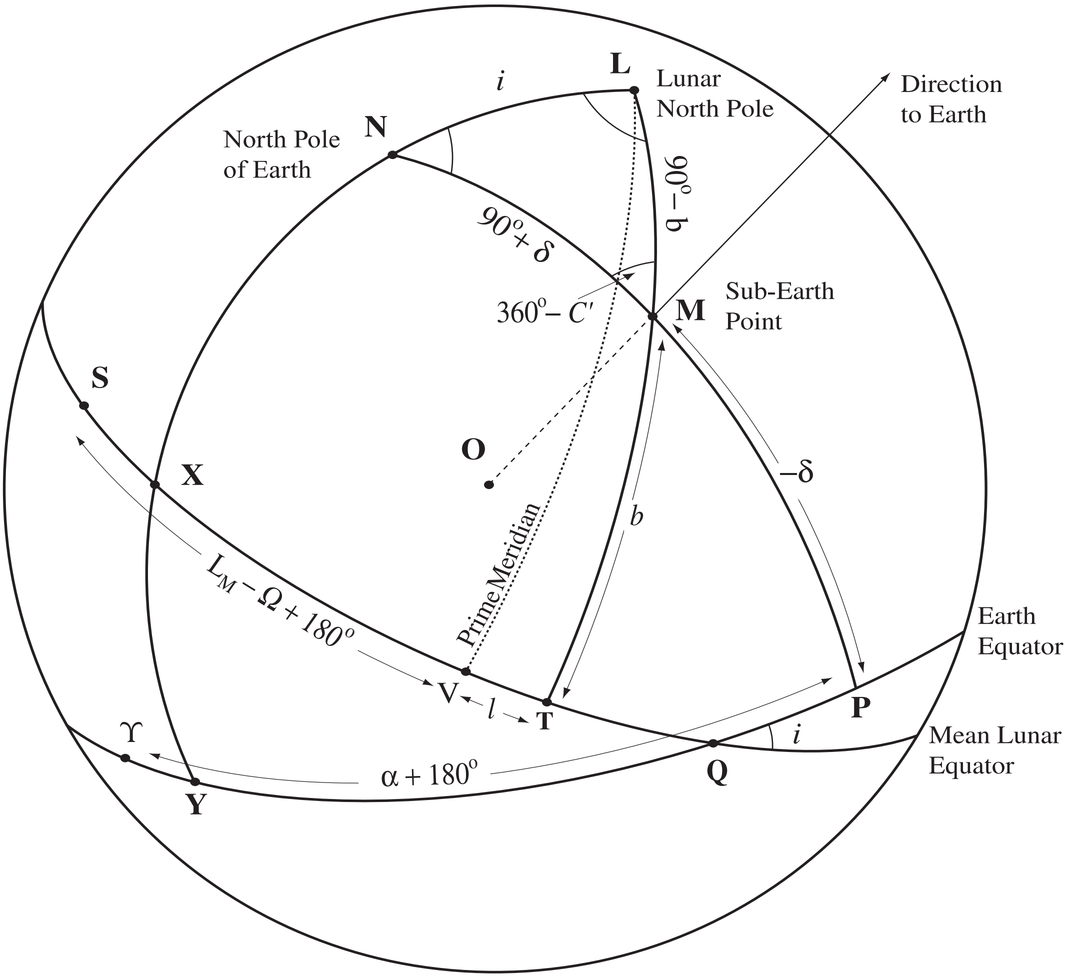

Figure 3: The selenocentric sphere: showing the relationships between the sub-Earth point M, *the mean lunar equator and the Earth's equator.*

3. **Implementation of JPL lunar rotation angles**

The JPL DE403/LE403 ephemeris, commonly known as DE403, includes an ephemeris for the rotation of the Moon. The problem is to transform these rotation angles into variables equivalent to the quantities Ω, I, L_M and hence use equations (8), (9), (10), and (18), (19) (or (20), (21)) with these newly determined quantities to compute librations l_T, b_T and position angle C'_T. We refer to l_T, b_T as the true or total lunar librations.

The Euler angles describing the rotation of the Moon, ϕ, θ and ψ are determined in Newhall and Williams (1996). These angles are defined relative to the ICRS Earth equator and equinox. They describe the orientation of the principal axes of inertia of the Moon (the PA system), sometimes called the axes of figure system. We show how they can be transformed to give new Euler angles ϕ_C, θ_C and ψ_C, which are defined relative to the ecliptic and equinox reference frame of date. These transformed angles are used to describe the orientation of a slightly different lunar axis system, which has one axis towards the mean Earth direction, and another along the rotation axis (the ME system), sometimes called the mean Earth/rotation axis system. From these quantities we can compute new Ω, I and L_M.

The Euler angles ϕ, θ, ψ and new set ϕ_C, θ_C, ψ_C are shown in Figures 4a, 4b respectively. These are given in Newhall and Williams (1996) and are included in this technical note for convenience.

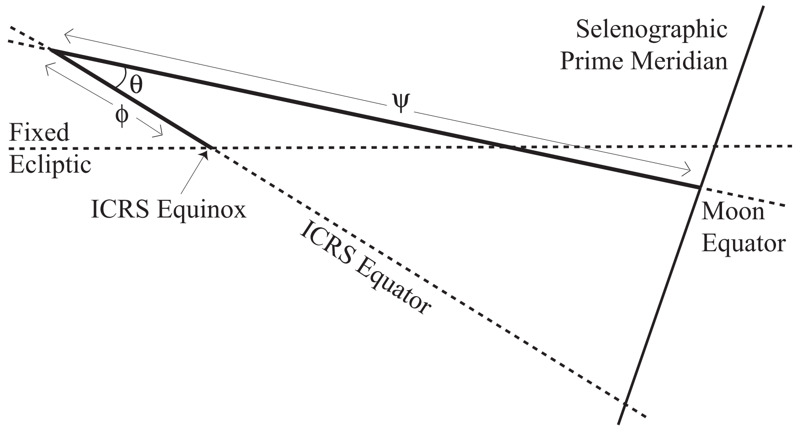

Figure 4a: Equatorial reference frame showing the Euler angles ϕ, θ, ψ, used to describe the lunar principal axis (PA) system. The value of ϕ shown in this diagram is negative.

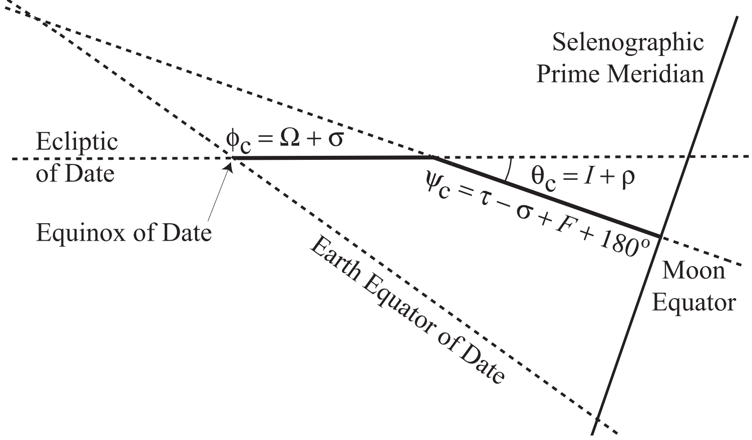

Figure 4b: Ecliptic reference frame showing the transformed Euler angles ϕ_C, θ_C, ψ_C, used to describe the orientation of the lunar ME system.

The angles are defined as:

ϕ = the angle along the ICRS equator, from the ICRS X-axis to the ascending node of the lunar equator;

θ = the inclination of the lunar equator to the ICRS equator;

ψ = the angle along the lunar equator from the node to the lunar prime meridian;

ϕ_C = the angle from the equinox of date to the descending node of the lunar equator on the ecliptic of date;

θ_C = the inclination of the lunar equator to the ecliptic of date;

ψ_C = the angle along the lunar equator from its descending node on the ecliptic to the lunar prime meridian.

From Seidelmann et al. (2007) a vector **p** in the PA system can be transformed to a vector **q** in the ME system by applying three small rotations. This expression (Konopliv et al. (2001)) is

$$\mathbf{q} = \mathbf{R}_1(-0.''1462)\ \mathbf{R}_2(-79.''0768)\ \mathbf{R}_3(-63.''8986)\ \mathbf{p} \tag{22}$$

where $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{R}_3$ are given in equations (4), (5), (6) respectively. It must be noted that the numerical values for the rotations in equation (22) are specific to DE403 and are different for other ephemerides.

The JPL lunar libration ephemeris gives the orientation of the PA system relative to the ICRS. So if we know the components of any vector in the PA system, then the JPL libration angles enable us to refer it to the ICRS. However the commonly-used reference system for lunar cartography is the ME system. Using the inverse of equation (22) we can convert any vector in the ME system into the PA system, and then we can use the JPL libration angles to convert to the ICRS. Finally we apply frame bias, precession and nutation, and convert to the ecliptic reference frame of date. We apply this process to vectors in the ME system that define the lunar pole and the lunar prime meridian, and use the converted vectors to calculate lunar libration angles for the ME system in the ecliptic-of-date frame.

Let $\mathbf{r}^{\mathrm{sel}}$ be a vector in the ME system. The vector $\mathbf{r}_1$ in the PA system is found from the inverse of equation (22)

$$\begin{aligned}\mathbf{r}_1 &= {\mathbf{R}_3}^{-1}(-63.''8986)\ {\mathbf{R}_2}^{-1}(-79.''0768)\ {\mathbf{R}_1}^{-1}(-0.''1462)\ \mathbf{r}^{\mathrm{sel}}\\ &= \mathbf{R}_3(63.''8986)\ \mathbf{R}_2(79.''0768)\ \mathbf{R}_1(0.''1462)\ \mathbf{r}^{\mathrm{sel}}.\end{aligned} \tag{23}$$

In the ICRS equator system (see Figure 4(a)) this vector becomes

$$\mathbf{r}_2 = \mathbf{R}_3(-\phi)\ \mathbf{R}_1(-\theta)\ \mathbf{R}_3(-\psi)\ \mathbf{r}_1. \tag{24}$$

Finally, in the ecliptic of date system this vector becomes

$$\mathbf{r}^{\mathrm{date}} = \mathbf{R}_1(\varepsilon)\ \mathbf{N}\ \mathbf{P}\ \mathbf{B}\ \mathbf{r}_2 \tag{25}$$

where **B**, **P** and **N** are the frame bias, precession and nutation matrices respectively, and $\mathbf{R}_1(\varepsilon)$ is the rotation to the true ecliptic of date reference frame. Note that **B**, the frame bias matrix, is included for completeness, as its effect is well under $0.''1$.

Now apply rotations in equations (23), (24), (25) to $\mathbf{r}^{\mathrm{sel}} = (1, 0, 0)$ and then to $\mathbf{r}^{\mathrm{sel}} = (0, 0, 1)$ and call the resulting vectors $\mathbf{x}^{\mathrm{date}}$ and $\mathbf{z}^{\mathrm{date}}$ respectively; they are with respect to the ecliptic of date system.

In the ecliptic of date system let **i**, **j**, **k** be unit vectors along the O_x, O_y and O_z axes respectively. Define the unit vector $\boldsymbol{\Omega}$ to be in the direction O to the descending node of the lunar equator on the ecliptic (S). We have

$$\boldsymbol{\Omega} = \frac{\mathbf{z}^{\mathrm{date}} \times \mathbf{k}}{|\ \mathbf{z}^{\mathrm{date}} \times \mathbf{k}\ |} \tag{26}$$

The angles ϕ_C, θ_C and ψ_C are found, using Figure 4(b), from the following formulae

$$\cos\phi_C = \mathbf{i}.\boldsymbol{\Omega} \tag{27}$$

$$\sin\phi_C = \mathbf{j}.\boldsymbol{\Omega} \tag{28}$$

$$\cos\theta_C = \mathbf{k}.\mathbf{z}^{\mathrm{date}} \tag{29}$$

$$\cos\psi_C = \boldsymbol{\Omega}.\mathbf{x}^{\mathrm{date}} \tag{30}$$

$$\sin\psi_C = (\mathbf{z}^{\mathrm{date}} \times \boldsymbol{\Omega}).\mathbf{x}^{\mathrm{date}}. \tag{31}$$

From Newhall and Williams (1996)

$$\phi_C = \mathit{\Omega} + \sigma \tag{32}$$

$$\theta_C = I + \rho \tag{33}$$

$$\psi_C = \tau - \sigma + F + 180°. \tag{34}$$

Since (see ES (1961) p.107)

$$F = L_M - \Omega \tag{35}$$

equation (34) becomes using (35)

$$\psi_C = \tau - \sigma + L_M - \Omega + 180°. \tag{36}$$

Substituting now for Ω from equation (32) into (36) we have

$$\psi_C = \tau - \sigma + L_M - \phi_C + \sigma + 180°$$

$$\psi_C + \phi_C - 180° = L_M + \tau. \tag{37}$$

From equations (32), (33) and (37) the changes in Ω, I and L_M viz. σ, ρ and τ respectively are as a result of the physical libration. To obtain the total or true librations l_T, b_T we must therefore substitute ϕ_C, θ_C and $\psi_C + \phi_C - 180°$ for Ω, I and L_M respectively in equations (8), (9) and (10). We note since the values for Ω and L_M that are substituted into equations (8), (9) and (10) are referred to the true equinox we must set $N = 0$. The position angle C'_T for the total librations is then found using equations (18), (19) or (20), (21).

We then can calculate the physical librations δl_P, δb_P and $\delta C'_P$ from

$$\delta l_P = l_T - l_O \tag{38}$$

$$\delta b_P = b_T - b_O \tag{39}$$

$$\delta C'_P = C'_T - C'_O \tag{40}$$

where the optical librations l_O, b_O and the position angle C'_O are as computed from equations in section 2.

The heliocentric ecliptic longitude, λ_H, and latitude, β_H, of the Moon are determined by calculating the vectors for the geocentric ecliptic positions of the Sun and Moon and forming the heliocentric vector to the Moon $(X, Y, Z)_{SM}$ and its length, d_{SM}.

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_M = d \begin{pmatrix} \cos\lambda\cos\beta \\ \sin\lambda\cos\beta \\ \sin\beta \end{pmatrix} \tag{41}$$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_S = d_S \begin{pmatrix} \cos\lambda_S\cos\beta_S \\ \sin\lambda_S\cos\beta_S \\ \sin\beta_S \end{pmatrix} \tag{42}$$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_{SM} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_M - \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_S \tag{43}$$

$$d_{SM} = \sqrt{X_{SM}^2 + Y_{SM}^2 + Z_{SM}^2} \tag{44}$$

$$\tan\lambda_H = \frac{Y_{SM}}{X_{SM}} \tag{45}$$

$$\sin\beta_H = \frac{Z_{SM}}{d_{SM}} \tag{46}$$

By substituting λ_H for λ and β_H for β in equations (8) and (9), the selenographic longitude of the Sun, l_S, can be determined in a similar manner to the libration in longitude. Similarly, substitution of these quantities into equation (10) allows the determination of the selenographic latitude, b_S, in a similar manner to the libration in latitude. The selenographic colongitude of the Sun is simply $90° - l_S$, adjusted to lie in the range $0°$ to $360°$.

4. **Numerical Example**

The purpose of this numerical example is to calculate the quantities on the odd pages D7–D21 of the AsA, most of which involve the lunar librations. The quantities tabulated are:

- The Earth's selenographic longitude (l_T) and latitude (b_T), which correspond to the total librations in longitude and latitude, respectively.
- The position angle of the axis of rotation (C'_T).
- The physical librations, δl_P, δb_P and the difference $\delta C'_P$, tabulated in thousandths of a degree.
- The Sun's selenographic colongitude ($90° - l_S$) and latitude (b_S).
- The geometrical quantities fraction illuminated (f_i) and the position angle of the bright limb (PA_B).

4.1 Algorithm for calculating the librations and position angle of the axis

The method used involves two passes through a common process. The first pass calculates the optical librations, which are geometrical in nature and adhere to Cassini's laws. The second pass calculates the total librations which includes the adopted rotational ephemeris of the Moon. In this example the JPL DE403 Ephemeris is used (see Section 3). IAU Standards of Fundamental Astronomy routines (SOFA) are used for obtaining the frame bias, precession and nutation. The steps are given below.

Step 1 Obtain the apparent positions α, δ, d, and α_s, δ_s, d_s for the Moon and Sun, respectively, at time t.

Step 2 Obtain the nutation in longitude ($\Delta\psi$) and obliquity ($\Delta\varepsilon$) and the true obliquity (ε) for time t.

Step 3 Determine the apparent ecliptic positions λ, β, and λ_s, β_s, by rotating the apparent equatorial coordinates around the X-axis by the angle ε.

Step 4 Calculate the light time correction for the Moon, $\tau = d/c$, where $c = 173{\cdot}1446\ 3268\ 467$ is the speed of light in au/day.

Step 5 Adopt the Newhall and Williams (1996) value of the inclination of the ecliptic to the mean lunar equator I of $5553''\!.6$.

Step 6 At time $t - \tau$ evaluate the fundamental arguments Ω and L_M using Simon et al (1994).

Step 7 Using equations (8), (9) and (10), calculate the optical (geometric) librations l and b from $\lambda - \Omega - N$, β, I and L_M, where $N = \Delta\psi$ is the nutation in longitude.

Step 8 Next, in order to determine C', the position angle of the axis of rotation, determine Ω' from equations (14) and (15), and i from equation (13). Thus C' may be calculated from equations (18) and (19).

Step 9 Now use the ephemeris and equations (27)-(31) to determine the Euler angles ϕ_C, θ_C and ψ_C. This process includes the transformations from the system of the ephemeris (JPL DE403) to the true of date system equations (23)-(25).

Step 10 Then substitute ϕ_C for Ω, θ_C for I and $\psi_C + \phi_C - 180°$ for L_M and repeat steps (7), (8) and (9) resulting this time in the total librations l_T, b_T and C'_T.

Step 11 Lastly the physical librations, the differences between the total and the optical librations are given by equations (38)-(40).

4.2 Numerical Example

Calculate the quantities described in this Technical Note for 2011 June 01 at 0^{h} TT, approximately 21 hours before the instant of new Moon.

$$t = 2455713{\cdot}5000000 \quad \text{2011 June 01 } 0^{\mathrm{h}} \text{ TT}$$

Step 1: The apparent positions of the Moon and Sun have been calculated using methods described in Section B of *The Astronomical Almanac*.

Moon	α	=	$57\overset{\circ}{.}364896851$	
	δ	=	$22\overset{\circ}{.}200527037$	
	d	=	$0{\cdot}0026441632$	au
Sun	α_S	=	$68\overset{\circ}{.}564159796$	
	δ_S	=	$21\overset{\circ}{.}975380381$	
	d_S	=	$1{\cdot}0139593548$	au

Step 2: Nutations and the mean and true obliquity have been calculated using SOFA routines.

$$N = \Delta\psi = 0\overset{\circ}{.}004500032$$
$$\Delta\varepsilon = -0\overset{\circ}{.}000366339$$
$$\varepsilon_m = 23\overset{\circ}{.}437794624$$
$$\varepsilon = \varepsilon_m + \delta\varepsilon = 23\overset{\circ}{.}437428285$$

Step 3: Apparent ecliptic positions

$$\begin{array}{lrcr} \text{Moon} & \lambda & = & 60^\circ\!.023691900 \\ & \beta & = & 2^\circ\!.094854205 \\ \text{Sun} & \lambda_S & = & 70^\circ\!.189728559 \\ & \beta_S & = & -0^\circ\!.000031006 \end{array}$$

Step 4: Light time (to first order), and JD corrected for light time,

$$\begin{array}{rcr} \tau & = & 0{\cdot}0000153\text{days} \\ t - \tau & = & 2455713{\cdot}4999847 \end{array}$$

Step 5: Adopting the mean inclination (I) from Newhall and Williams (1996).

$$I = 1^\circ\!.542666667$$

Step 6: Fundamental arguments,

$$\begin{array}{rcr} \Omega & = & 264^\circ\!.306813985 \\ L_M & = & 424^\circ\!.125125229 \end{array}$$

Step 7: Optical librations

$$\begin{array}{rcr} l_O & = & -4^\circ\!.046692371 \\ b_O & = & -2^\circ\!.728684824 \end{array}$$

Step 8: Optical position angle of the axis of rotation and associated intermediate quantities.

$$\begin{array}{rcr} C'_O & = & 346^\circ\!.197699892 \\ \Omega' & = & 3^\circ\!.830995947 \\ i & = & 23^\circ\!.637422107 \\ \Delta & = & 80^\circ\!.798845156 \end{array}$$

Step 9: Interrogate the JPL DE403/LE403 lunar ephemeris to obtain the three Euler angles ϕ, θ and ψ or PV(1), PV(2) and PV(3) respectively if the JPL Fortran routine PLEPH is used.

Please note that these quantities are given here in radians i.e. as they are provided by this routine.

$$\begin{array}{rcr} \phi & = & 0{\cdot}067143410 \\ \theta & = & 0{\cdot}412412621 \\ \psi & = & 3522{\cdot}780883138 \end{array}$$

The matrix to transform the vector $\mathbf{r}^{\text{sel}}$ to $\mathbf{r_1}$, equation (23) is:

$$\mathbf{R}_3(63''\!.8986)\mathbf{R}_2(79''\!.0768)\mathbf{R}_1(0''\!.1462) = \begin{pmatrix} +0{\cdot}999999879 & +0{\cdot}000309789 & -0{\cdot}000383375 \\ -0{\cdot}000309789 & +0{\cdot}999999952 & +0{\cdot}000000828 \\ +0{\cdot}000383375 & -0{\cdot}000000709 & +0{\cdot}999999927 \end{pmatrix}$$

The rotation matrix required to transform the vector $\mathbf{r}_1$ to $\mathbf{r}_2$ (see equation (24)):

$$\mathbf{R}_3(-\phi)\mathbf{R}_1(-\theta)\mathbf{R}_3(-\psi) = \begin{pmatrix} -0{\cdot}438179850 & +0{\cdot}898484961 & +0{\cdot}026892261 \\ -0{\cdot}828464431 & -0{\cdot}392061910 & -0{\cdot}399917672 \\ -0{\cdot}348776583 & -0{\cdot}197515147 & +0{\cdot}916156461 \end{pmatrix}$$

Use SOFA routines to generate $\mathbf{N\,P\,B}$ (equation (25)) and $\mathbf{R}_1\,(\varepsilon)$ that are required to transform the vector $\mathbf{r}_2$ to $\mathbf{r}_{\text{date}}$:

$$\mathbf{N\,P\,B} = \begin{pmatrix} +0{\cdot}999995907 & -0{\cdot}002624146 & -0{\cdot}001140060 \\ +0{\cdot}002624153 & +0{\cdot}999996557 & +0{\cdot}000004945 \\ +0{\cdot}001140043 & -0{\cdot}000007937 & +0{\cdot}999999350 \end{pmatrix}$$

$$\mathbf{R}_1(\varepsilon) = \begin{pmatrix} +1{\cdot}000000000 & +0{\cdot}000000000 & +0{\cdot}000000000 \\ +0{\cdot}000000000 & +0{\cdot}917494994 & +0{\cdot}397747326 \\ +0{\cdot}000000000 & -0{\cdot}397747326 & +0{\cdot}917494994 \end{pmatrix}$$

Thus the vectors $\mathbf{x}^{\text{date}}$ and $\mathbf{z}^{\text{date}}$ are:

$$\mathbf{x}^{\text{date}} = (\,-0{\cdot}435874783 \quad -0{\cdot}899952706 \quad +0{\cdot}009914620\,)$$
$$\mathbf{z}^{\text{date}} = (\,+0{\cdot}027064863 \quad -0{\cdot}002095582 \quad +0{\cdot}999631483\,)$$

Step 9 continued: Thus the Euler angles in the required reference system are:

$$\begin{aligned} \phi_C &= 265^{\circ}.572527636 \\ \theta_C &= 1^{\circ}.555534881 \\ \psi_C &= 338^{\circ}.577958345 \end{aligned}$$

Step 10: Repetition of steps 7, 8 and 9 giving the total librations and position angle

$$\begin{aligned} \psi_C + \theta_C - 180^{\circ} &= 64^{\circ}.150485981 \\ \Omega' &= 3^{\circ}.875459322 \\ i &= 23^{\circ}.605632357 \\ \Delta &= 82^{\circ}.018859987 \\ l_T &= -4^{\circ}.067219698 \\ b_T &= -2^{\circ}.765029585 \\ C'_T &= 346^{\circ}.200360493 \end{aligned}$$

Step 11: Physical librations and difference in position angle

$$\begin{aligned} \delta l_P = l_T - l_O &= -0^{\circ}.020527328 \\ \delta b_P = b_T - b_O &= -0^{\circ}.036344761 \\ \delta C'_P = C'_T - C'_O &= 0^{\circ}.002660602 \end{aligned}$$

The remainder are the other quantities published on AsA Section D.

Heliocentric ecliptic coordinates of the Moon: The heliocentric ecliptic longitude, λ_H, and latitude, β_H, of the Moon are determined by calculating the vectors for the geocentric ecliptic positions of the Sun and Moon and forming the heliocentric vector to the Moon. From these quantities the spherical coordinates λ_H and β_H are

$$\begin{aligned} \lambda_H &= 250^{\circ}.216150415 \\ \beta_H &= 0^{\circ}.005506792 \end{aligned}$$

Selenographic coordinates of the Sun: The selenographic longitude, l_S, latitude, b_S and colongitude, $90^{\circ} - l_S$, of the Sun are determined using equations (8), (9) and (10) and substituting λ_H and β_H for the ecliptic longitude and latitude λ, β respectively.

$$\begin{aligned} l_S &= 186^{\circ}.070912360 \\ b_S &= 0^{\circ}.406387923 \\ 90^{\circ} - l_S &= 263^{\circ}.929087640 \end{aligned}$$

Moon orientation and illumination: Calculate PA_B, the position angle of the bright limb of the Moon from equation (C.4) and the fraction illuminated from (C.7).

$$\begin{aligned} E &= 10^{\circ}.377412659 \\ \cos(E_S) &= -0{\cdot}983557618 \\ PA_B &= 89^{\circ}.127532454 \\ f_i &= 0{\cdot}008221191 \end{aligned}$$

In the following summary the tabular quantities from the putative right-hand page D13 for JD 245 5713·5 (2011 June 1) for both the DE403-based data and the Eckhardt-based data are given.

	DE403	Eckhardt		DE403	Eckhardt
l_T	−4·067	−4·068	$90^{\circ} - l_S$	263·93	263·93
b_T	−2·765	−2·742	b_S	0·41	0·38
δl_P	−0·021	−0·021	C'_T	346·200	346·201
δb_P	−0·036	−0·013	PA_B	89·13	89·13
$\delta C'_P$	+0·003	+0·003	f_i	0·008	0·008

It can be seen that there is a difference of $-0^{\circ}.023$ in the physical libration in latitude in the sense DE403 minus Eckhardt. This propagates into both the total libration in latitude (Earth's selenographic latitude) and the Sun's selenographic latitude. Users comparing data from *The Astronomical Almanac* in the period 1985 to 2010 will find similar differences. An explanation of this difference requires further study of the Eckhardt model and is beyond the scope of this paper. However, it does merit further investigation.

5. **Acknowledgements**

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We would like to thank Julia Weratschnig for her translation from the German of Encke's paper (1843). We also thank John Irwin for pointing out our erroneous adjustment of the inclination of the ecliptic to the mean lunar equator by the nutation in obliquity.

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Appendix A

Estimation of the magnitude of the optical lunar librations

Let L_L denote the longitude of the Moon measured along the lunar equator from the ascending node of the Moon on the ecliptic. We can relate L_L and b to the orbital elements of the Moon. We have

$$\sin b = \sin(\gamma + I)\sin(\omega + f) \tag{A.1}$$

$$\tan L_L = \cos(\gamma + I)\tan(\omega + f) \tag{A.2}$$

where γ is the inclination of the Moon's orbit to the ecliptic and ω, f are the argument of pericentre and true anomaly of the Moon. γ is at most about $5\frac{1}{2}°$ and I is about $1\frac{1}{2}°$. We can write (A.2) as

$$L_L = \omega + f + \sum_{s=1}^{\infty} \frac{(-1)^s}{s} \tan^{2s}(\frac{(\gamma + I)}{2}) \sin 2s(\omega + f). \tag{A.3}$$

As the largest term under the summation in (A.3) is of the order of 0·2° we can write

$$\begin{aligned} L_L &\simeq \omega + f \\ &\simeq \omega + M + 2e\sin M + \frac{5}{4}e^2 \sin 2M + O(e^3) \\ &\simeq L_M - \Omega + 2e\sin M + \frac{5}{4}e^2 \sin 2M + O(e^3) \end{aligned} \tag{A.4}$$

where e is the Moon's eccentricity, about 0·05, and M is the mean anomaly for the Moon.

From (A.1) b can be at most $\gamma + I$, that is about $\pm 7°$. From equation (A.4) we see that the variations of the Moon's longitude measured along the lunar equator from its mean longitude minus the node $(L_M - \Omega)$ are primarily due to the eccentricity (with the largest term of amplitude about 6°). These contribute to the leading terms in the libration in longitude (l).

Appendix B

Approximate method to calculate the lunar libration angles

To prove, in the notation of section 2, that the lunar librations can be computed from the approximate formulae

$$l = \lambda + \mu + Ab - (L_M + N) \tag{B.1}$$
$$b = B - \beta \tag{B.2}$$

where the auxiliary variables A,B and μ are defined by

$$A = \sin I \cos(\lambda - \Omega) \tag{B.3}$$
$$\tan B = -\tan I \sin(\lambda - \Omega) \tag{B.4}$$
$$\sin \mu = \tan^2 \frac{I}{2} \sin 2(\lambda - \Omega). \tag{B.5}$$

In using this approximate method the auxiliaries A, B and μ were tabulated against the argument $\lambda - \Omega$.

These formulae were given in the ES (1961) on page 319 without details of their derivation. The steps given here for their proof uses the method described in Encke (1843).

In section 2 the point M in Figure 1 in the ecliptic system O_{xyz} is related to this point in the mean equator of the Moon system $O_{x'y'z'}$ by equation (7). This leads to the equations (8), (9), (10) with $N = 0$, which we give again here for convenience,

$$\cos b \cos(l + L_M - \Omega) = \cos \beta \cos(\lambda - \Omega) \tag{B.6}$$
$$\cos b \sin(l + L_M - \Omega) = \cos I \cos \beta \sin(\lambda - \Omega) - \sin I \sin \beta \tag{B.7}$$
$$\sin b = -\sin I \cos \beta \sin(\lambda - \Omega) - \cos I \sin \beta. \tag{B.8}$$

Consider point M^* in Figure 1 and let UM^* be B and VM^* be l^*. Then this point in the ecliptic system is

$$\begin{pmatrix} \cos B \cos(\lambda + 180° - \Omega) \\ \cos B \sin(\lambda + 180° - \Omega) \\ -\sin B \end{pmatrix} \tag{B.9}$$

and in the mean equatorial system of the Moon

$$\begin{pmatrix} \cos(L_M + 180° - \Omega + l^*) \\ \sin(L_M + 180° - \Omega + l^*) \\ 0 \end{pmatrix} \tag{B.10}$$

and they will be related by

$$\begin{pmatrix} \cos(L_M + 180° - \Omega + l^*) \\ \sin(L_M + 180° - \Omega + l^*) \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos I & -\sin I \\ 0 & \sin I & \cos I \end{pmatrix} \begin{pmatrix} \cos B \cos(\lambda + 180° - \Omega) \\ \cos B \sin(\lambda + 180° - \Omega) \\ -\sin B \end{pmatrix} \tag{B.11}$$

giving the equations

$$\cos(l^* + L_M - \Omega) = \cos B \cos(\lambda - \Omega) \tag{B.12}$$
$$\sin(l^* + L_M - \Omega) = \cos I \cos B \sin(\lambda - \Omega) - \sin I \sin B \tag{B.13}$$
$$0 = \sin I \cos B \sin(\lambda - \Omega) + \cos I \sin B. \tag{B.14}$$

The first step is using equations (B.12), (B.13) and (B.14) we can write equations (B.6), (B.7) and (B.8) as

$$\cos b \sin(l - l^*) = \sin I \cos(\lambda - \Omega) \sin(B - \beta) \tag{B.15}$$
$$\cos b \cos(l - l^*) = \cos(B - \beta) \tag{B.16}$$
$$\sin b = \frac{\cos I}{\cos B} \sin(B - \beta). \tag{B.17}$$

From equations (B.12), (B.13) and (B.14) we have

$$\tan(l^* + L_M - \Omega) = \tan(\lambda - \Omega)\sec I \tag{B.18}$$

$$\tan B = -\sin(\lambda - \Omega)\tan I. \tag{B.19}$$

Define,

$$A = \cos(\lambda - \Omega)\sin I \tag{B.20}$$

$$D = \frac{\cos I}{\cos B} \tag{B.21}$$

then from equations (B.15), (B.16), (B.17) using (B.20), (B.21) we have

$$\tan(l - l^*) = A\tan(B - \beta) \tag{B.22}$$

$$\sin b = D\sin(B - \beta). \tag{B.23}$$

We can write equation (B.18) as

$$l^* + L_M = \lambda + \tan^2\frac{I}{2}\sin 2(\lambda - \Omega) + \frac{1}{2}\tan^4\frac{I}{2}\sin 4(\lambda - \Omega) + O(\tan^6\frac{I}{2}). \tag{B.24}$$

This is obtained by applying the method of expansion described in Brouwer and Clemence (1961), Chapter II, section 3 to equation (B.18).

From equations (B.16) and (B.17) we have

$$\tan b = \frac{\cos I}{\cos B}\tan(B - \beta)\cos(l - l^*) \tag{B.25}$$

from which we obtain to a good approximation

$$b = B - \beta. \tag{B.26}$$

This follows from noting B can never be larger than I and using again the method of expansion in Brouwer and Clemence, Chapter II, section 3. The largest difference between $B - \beta$ and b will be

$$\tan^2\frac{I}{2}\sin 2(B - \beta). \tag{B.27}$$

The ecliptic latitude β has a maximum value of about $5\frac{1}{2}^\circ$ (see *The Astronomical Almanac* for the year 2010, page D22) and B at most can be I (see Figure 1). Thus taking I as $5553.''6 = 1^\circ\ 32'\ 33.''6$ (Newhall and Williams (1996)) the largest value of $B - \beta$ is about 7° so term (B.27) has a maximum value of about $9''$.

We write equation (B.24) as

$$l^* = \lambda - L_M + \sin\mu + \frac{1}{2}\tan^4\frac{I}{2}\sin 4(\lambda - \Omega) + O(\tan^6\frac{I}{2}) \tag{B.28}$$

where we have set

$$\sin\mu = \tan^2\frac{I}{2}\sin 2(\lambda - \Omega). \tag{B.29}$$

Next we note we can write (B.22) to a good approximation as

$$l - l^* = A(B - \beta). \tag{B.30}$$

The largest term neglected being about $3''$.

Substituting l^* from equation (B.30) into (B.28) we obtain

$$l = \lambda - L_M + \sin\mu + \frac{1}{2}\tan^4\frac{I}{2}\sin 4(\lambda - \Omega) + A(B - \beta) + O(\tan^6\frac{I}{2}). \tag{B.31}$$

From equation (B.29) we see as μ is small we can write $\sin\mu \simeq \mu$ and using (B.26) we can write equation (B.31) as

$$l = \lambda - L_M + \mu + Ab + \frac{1}{2}\tan^4\frac{I}{2}\sin 4(\lambda - \Omega) + O(\tan^6\frac{I}{2}). \tag{B.32}$$

As λ is referred to the true equinox and L_M to the mean equinox, in equation (B.32) we must add N the nutation in longitude to L_M. We have then

$$l = \lambda + \mu + Ab - (L_M + N) + \frac{1}{2}\tan^4\frac{I}{2}\sin 4(\lambda - \Omega) + O(\tan^6\frac{I}{2}). \tag{B.33}$$

The librations l, b are found from equations (B.33) and (B.26) where the auxiliary variables A, B and μ are defined in equations (B.20), (B.19) and (B.29) respectively. Dropping in l the small terms of order $\tan^4\frac{I}{2}$ and higher order in $\tan\frac{I}{2}$ then these are the approximate formulae for calculating the lunar librations.

Appendix C

Geometric Quantities provided on the Moon's Physical Ephemeris Pages

For the Moon the position angle of the bright limb and the fraction illuminated are calculated in the following manner. They are purely geometric quantities tabulated on the physical ephemeris pages of Section D and their determination is provided here for the sake of completeness.

The position angle of the bright limb, PA_B, is the position angle of the midpoint of the illuminated fraction of the Moon measured eastward from the north point of the disk. It is also the position angle of the subsolar point. It should be noted that it is *not* measured from the direction of the observer's zenith. The zenith angle of the bright limb is PA_B minus the parallactic angle of the Moon. PA_B can be calculated from the spherical triangle PSM formed on the celestial sphere by the north celestial pole, the Sun and the Moon (see Figure 5). E is the geocentric elongation of the Moon from the Sun, α and δ are the apparent right ascension and declination of the Moon and α_S and δ_S are the same quantities for the Sun respectively.

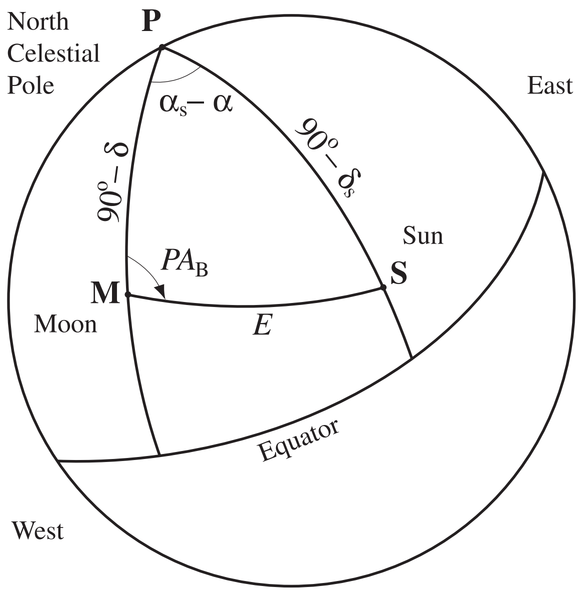

Figure 5: Diagram showing the spherical triangle PSM that specifies the illuminated limb of the Moon (PA_B), the north celestial pole and the apparent equatorial positions of the Sun and Moon.

Using the spherical triangle PSM shown in Figure 5, the following relations may be written down

$$\sin E \sin PA_B = \cos\delta_S \sin(\alpha_S - \alpha) \tag{C.1}$$

$$\sin E \cos PA_B = \sin\delta_S \cos\delta - \cos\delta_S \sin\delta \cos(\alpha_S - \alpha) \tag{C.2}$$

$$\cos E = \sin\delta_S \sin\delta + \cos\delta_S \cos\delta \cos(\alpha_S - \alpha) \tag{C.3}$$

Dividing equation (C.1) by equation (C.2) gives

$$\tan(PA_B) = \frac{\cos\delta_S \sin(\alpha_S - \alpha)}{\sin\delta_S \cos\delta - \cos\delta_S \sin\delta \cos(\alpha_S - \alpha)} \tag{C.4}$$

Ensure that PA_B is in the right quadrant and place in the range 0° to 360° by adding or subtracting multiples of 360°.

As seen from the Earth, the illuminated fraction of the Moon, f_i, can be calculated knowing the phase angle E_S (see Figure 6) and the ratio of the lengths TM to AM or the ratio of the illuminated portion of the lunar disk NTSM to its total area NASM (see Figure 7).

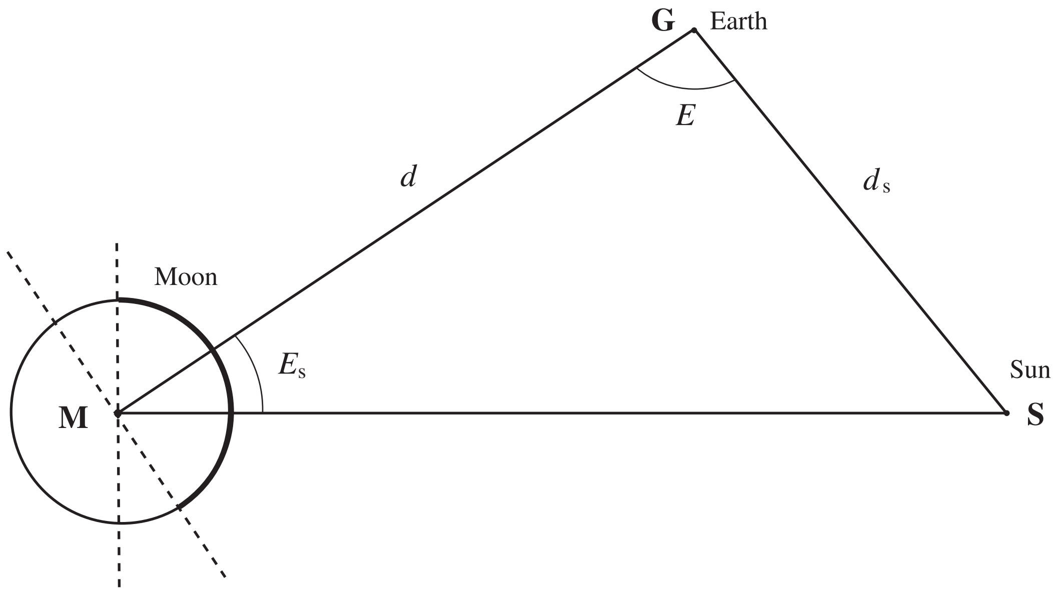

Figure 6: The plane triangle GMS showing the phase angle and the geometrical relationship between the Earth, Moon and Sun.

Using the plane triangle GMS in Figure 6, E_S may be calculated from

$$\sin E_S = d_S \sin E / \sqrt{(d^2 + d_S^2 - 2\,d\,d_S \cos E)} \tag{C.5}$$

$$\cos E_S = (d - d_S \cos E) / \sqrt{(d^2 + d_S^2 - 2\,d\,d_S \cos E)} \tag{C.6}$$

by taking the inverse tangent and ensuring it is in the correct quadrant, and E is the geocentric elongation of the Moon from the Sun given in equation (C.3).

Thus the ratio of the illuminated disk to the whole disk (TM to AM) can be expressed as

$$f_i = \frac{1}{2}(1 + \cos E_S) \tag{C.7}$$

where $\cos E_S$, the cosine of the phase angle, is calculated from equation (C.6), and f_i, takes values from 0 at New Moon to 1 at Full Moon.

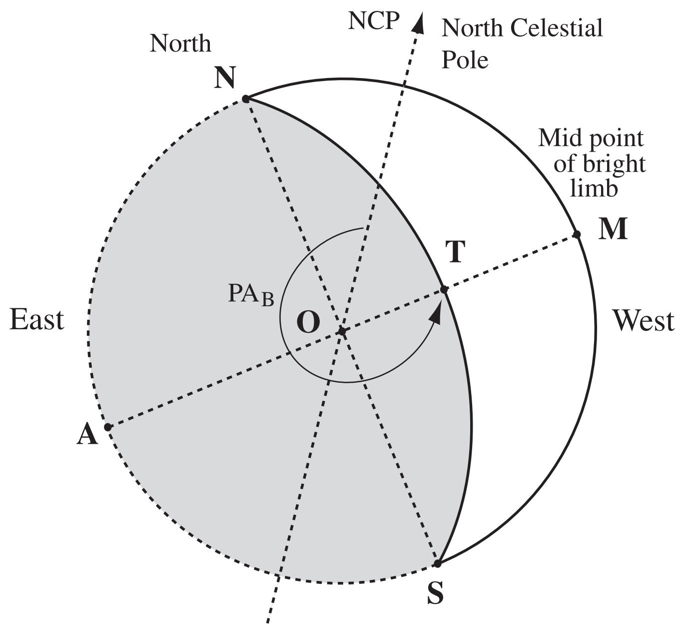

Figure 7: The observer's view of an approximately 4-day old Moon showing the position angle of its bright limb and the ratios used in determining the fraction illuminated.