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Determination of Starting Conditions for
Ephemerides of the Uranian Satellites I-V

by

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Summary

This technical note together with the paper Taylor (1998) shows clearly how starting conditions can be derived to produce ephemerides of the five major satellites of Uranus by numerical integration. It provides the formulae needed at various stages of the derivation which were omitted in Taylor (1998) for conciseness. From the equations of motion the variational equations are derived in component form suitable for programming purposes. Following a fit to observations of orbits based on simple precessing ellipses a first approximation is obtained to the starting conditions for a numerical integration. These are refined in the fit of the numerical integration to observations by initially iterating on osculating elements at epoch and then when the solution has converged to revert to position and velocity.

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The following two typographic errors have been corrected:

1) Equation B.14, an a has been inserted in the denominator of the 2nd term so that it reads $a\mu$.
2) Equation B.38, an a has been removed from the denominator.

Determination of starting conditions for ephemerides of the Uranian satellites I - V

D.B. Taylor

1. Introduction

Ephemerides of the five major Uranian satellites were developed in Taylor(1998) by numerical integration. In that paper references are given to all available Uranian satellite observations and an overview of the data is given. A brief description is made of the compilation of a catalogue of observations. A numerical integration was fitted to a subset of the data and results from this fit and further solutions made were discussed. An important part of this work was the determination of the initial conditions for the numerical integration. A description of the procedure used to obtain these starting conditions was given but the details and formulae used in the derivation were omitted for conciseness. In this technical note the formulae used are given so that this note together with the paper Taylor(1998) will show clearly how to determine these starting conditions.

In describing this procedure, at different stages fits of approximate orbits and then a numerical integration to data are made. The details of these least-squares fits are given in Taylor(1998) and are not repeated here. In sect. 2 the equations of motion are given. From these the variational equations are derived in component form suitable for programming purposes. In sect. 3 approximate theories are derived based on simple precessing ellipses with added perturbations arising from the Laplacian resonance. Expressions are derived for the satellite vectors following expansion in the eccentricity and inclination. The resulting elements of the theory from a fit to the observations are given and briefly discussed. In sect. 4 it is shown how using the approximate orbits a first estimate of starting conditions for a fit of a numerical integration to observations can be obtained. For the fit of the numerical integration to observations it is found necessary before iterating on rectangular coordinates and velocity components at epoch to iterate firstly on the osculating elements at epoch to obtain closer approximations to the starting conditions. In order to obtain osculating elements at epoch, in the equations of condition the partial derivatives computed from the variational equations must be transformed to partial derivatives with respect to orbital elements at epoch. Formulae for the partial derivatives of position and velocity with respect to orbital elements, necessary for this transformation, are given.

2. The equations of motion and variation

2.1 Equations of motion

The equatorial plane of Uranus is chosen as the reference plane for the numerical integration. This is the natural reference plane for the system, since the orbits have small inclinations to the equatorial plane, the largest being $\approx 4^\circ\!.3$ for Miranda. The calculation of the J_2 and J_4 perturbations are also simplified using this reference plane. It is assumed that the theoretical rate of precession of this reference plane is very small, and the Coriolis terms in the equations of motion due to using this non-inertial frame are negligible, certainly over the time-span of a few decades. The equations were formulated in Cartesian coordinates with the x-axis taken to be in the direction of the ascending node of the equator of Uranus on the Earth equator of B1950.0 (JED 2433282.423), the y-axis in the equatorial plane of Uranus, and the origin at the centre of mass of Uranus.

The equations of motion and variational equations are given for 5 mutually perturbing satellites, orbiting an oblate primary and perturbed by the Sun and 2 planets, Jupiter and Saturn. It should be noted these equations can easily be adapted for other satellite systems with different numbers of mutually perturbing satellites and perturbing planets. The equations of motion for the satellites are, for $i = 1, 2, \ldots 5$;

$$\ddot{\mathbf{r}}_i = \frac{-GM(1+m_i)\mathbf{r}_i}{r_i^3} + \sum_{\substack{j=1 \\ \neq i}}^{5} GMm_j \left(\frac{\mathbf{r}_j - \mathbf{r}_i}{r_{ij}^3} - \frac{\mathbf{r}_j}{r_j^3} \right)$$

$$+ GM_s\left(\frac{\mathbf{r}_s - \mathbf{r}_i}{r_{is}^3} - \frac{\mathbf{r}_s}{r_s^3}\right) + \sum_{p=1}^{2} GM_p\left(\frac{\mathbf{R}_p - \mathbf{r}_i}{R_{ip}^3} - \frac{\mathbf{R}_p}{R_p^3}\right)$$

$$+ A_i\mathbf{r}_i + B_i\hat{\mathbf{k}} + \sum_{\substack{l=1\\ \neq i}}^{5}(m_l A_l \mathbf{r}_l + m_l B_l \hat{\mathbf{k}}) \qquad (1)$$

where

$$A_i = \frac{GM}{r_i^3}\sum_{n=2}^{4} J_n \frac{R_u^n}{r_i^n} P'_{n+1}(\frac{z_i}{r_i}) \qquad (2)$$

$$B_i = \frac{-GM}{r_i^2}\sum_{n=2}^{4} J_n \frac{R_u^n}{r_i^n} P'_n(\frac{z_i}{r_i}) \ , \qquad (3)$$

with

$$P'_{n+1} = (n+1)P_n + P'_n \cdot z_i / r_i \cdot \qquad (4)$$

The subscripts $i = 1, 2 \ldots, 5$ have the usual convention of referring to satellites in order of increasing semi-major axis (i.e. 1=Miranda, 2=Ariel, 3=Umbriel, 4=Titania, 5=Oberon). The subscripts $p = 1, 2$ refer to planets Jupiter and Saturn respectively. We have

$\mathbf{r_i}$	position vector of satellite i
$\mathbf{r_s}$	position vector of the Sun relative to the primary
$\mathbf{R_p}$	position vector of the p^{th} planet relative to the primary
$\hat{\mathbf{k}}$	unit vector in the z-direction
r_i	distance of satellite i from the centre of the primary
z_i	the third component of the coordinates for the i^{th} satellite
r_s	distance of Sun from the centre of the primary
R_p	distance of p^{th} planet from the centre of the primary
r_{ij}	distance between satellites i and j
r_{is}	distance between satellite i and the Sun
R_{ip}	distance between satellite i and the p^{th} planet
G	gravitational constant
m_i	mass of satellite i divided by the mass of the primary
M	mass of Uranus
M_s	mass of the Sun
M_p	mass of the p^{th} planet
R_u	equatorial radius of the primary (26 200 km was used)
J_n	oblateness parameters
P'_n	derivative of the Legendre polynomial P_n.

The masses of the satellites are taken in units of the mass of the primary because it is expected that there will be lower correlation with the mass of the primary when taken in this form than if taken in absolute units.

The Solar and planetary perturbations were computed from the JPL DE200 ephemeris of Uranus, Jupiter and Saturn, referred to the Earth equator and equinox of B1950.0. The Solar and planetary coordinates were each represented in 400 day intervals by a Chebyshev series. Let (X_s, Y_s, Z_s) be the coordinates of the Sun relative to the primary. They are transformed to the integration reference plane by

$$\begin{pmatrix} x_s \\ y_s \\ z_s \end{pmatrix} = \begin{pmatrix} -\sin\alpha & \cos\alpha & 0 \\ -\cos\alpha\sin\delta & -\sin\alpha\sin\delta & \cos\delta \\ \cos\alpha\cos\delta & \sin\alpha\cos\delta & \sin\delta \end{pmatrix} \begin{pmatrix} X_s \\ Y_s \\ Z_s \end{pmatrix} \tag{5}$$

where α, δ are the B1950.0 right ascension and declination of the pole of the primary. The planetary perturbations in the integration reference plane are computed in a similar way; starting with planetary positions relative to the primary and applying the rotation in Eq. (5).

The oblateness accelerations have been modelled using formulae given in Merson and Odell (1975). The perturbation

$$\sum_{\substack{l=1\\ \neq i}}^{5} (m_l A_l \mathbf{r}_l + m_l B_l \hat{\mathbf{k}})$$

arises from the component of the attraction of each satellite on the primary caused by the oblateness of the primary.

2.2 **Variational equations**

Denote the components of $\mathbf{r}_i$,$\mathbf{r}_s$ and $\mathbf{R}_p$ by

$$\begin{aligned} \mathbf{r}_i &= (c_{3i-2}, c_{3i-1}, c_{3i}) \quad (i = 1, 2, \ldots, 5) \\ \mathbf{r}_s &= (x_s, y_s, z_s) \\ \mathbf{R}_p &= (x_p, y_p, z_p) \quad (p = 1, 2). \end{aligned}$$

In component form we can write Eq. (1) for $i = 1, 2, \ldots, 5$ as

$$\begin{aligned} \ddot{c}_{3i-2} &= F_{3i-2} \\ \ddot{c}_{3i-1} &= F_{3i-1} \\ \ddot{c}_{3i} &= F_{3i} \end{aligned} \tag{6}$$

where

$$\begin{aligned} F_{3i-2} = & \frac{-GM(1+m_i)c_{3i-2}}{r_i^3} \\ & + \sum_{\substack{j=1\\ \neq i}}^{5} GMm_j \left(\frac{c_{3j-2} - c_{3i-2}}{r_{ij}^3} - \frac{c_{3j-2}}{r_j^3} \right) \\ & + GM_s \left(\frac{x_s - c_{3i-2}}{r_{is}^3} - \frac{x_s}{r_s^3} \right) \\ & + \sum_{p=1}^{2} GM_p \left(\frac{x_p - c_{3i-2}}{R_{ip}^3} - \frac{x_p}{{R_p}^3} \right) \\ & + A_i c_{3i-2} + \sum_{\substack{l=1\\ \neq i}}^{5} m_l A_l c_{3l-2} \end{aligned} \tag{7}$$

$$
\begin{aligned}
F_{3i-1} = & \frac{-GM(1+m_i)c_{3i-1}}{r_i^3} \\
& + \sum_{\substack{j=1 \\ \neq i}}^{5} GMm_j \left(\frac{c_{3j-1} - c_{3i-1}}{r_{ij}^3} - \frac{c_{3j-1}}{r_j^3} \right) \\
& + GM_s \left(\frac{y_s - c_{3i-1}}{r_{is}^3} - \frac{y_s}{r_s^3} \right) \\
& + \sum_{p=1}^{2} GM_p \left(\frac{y_p - c_{3i-1}}{R_{ip}^3} - \frac{y_p}{{R_p}^3} \right) \\
& + A_i c_{3i-1} + \sum_{\substack{l=1 \\ \neq i}}^{5} m_l A_l c_{3l-1}
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
F_{3i} = & \frac{-GM(1+m_i)c_{3i}}{r_i^3} + \sum_{\substack{j=1 \\ \neq i}}^{5} GMm_j \left(\frac{c_{3j} - c_{3i}}{r_{ij}^3} - \frac{c_{3j}}{r_j^3} \right) \\
& + GM_s \left(\frac{z_s - c_{3i}}{r_{is}^3} - \frac{z_s}{r_s^3} \right) \\
& + \sum_{p=1}^{2} GM_p \left(\frac{z_p - c_{3i}}{R_{ip}^3} - \frac{z_p}{{R_p}^3} \right) \\
& + A_i c_{3i} + B_i + \sum_{\substack{l=1 \\ \neq i}}^{5} (m_l A_l c_{3l} + m_l B_l)
\end{aligned} \tag{9}
$$

and

$$
\begin{aligned}
r_i^2 &= c_{3i-2}^2 + c_{3i-1}^2 + c_{3i}^2 \\
r_{ij}^2 &= (c_{3j-2} - c_{3i-2})^2 + (c_{3j-1} - c_{3i-1})^2 + (c_{3j} - c_{3i})^2 \\
r_s^2 &= x_s^2 + y_s^2 + z_s^2 \\
r_{is}^2 &= (x_s - c_{3i-2})^2 + (y_s - c_{3i-1})^2 + (z_s - c_{3i})^2 \\
R_p^2 &= x_p^2 + y_p^2 + z_p^2 \\
R_{ip}^2 &= (x_p - c_{3i-2})^2 + (y_p - c_{3i-1})^2 + (z_p - c_{3i})^2.
\end{aligned} \tag{10}
$$

To fit the numerical integration to the observations we need the partial derivatives of each coordinate with respect to each solved-for parameter. These parameters will be the initial coordinates of the satellites, the initial velocities, $J_s(s = 2, 3, 4)$, the masses of the satellites and the mass of the primary; a total of 39 parameters which we denote by $q_1, q_2, \ldots, q_{39}$. In the fit to observations other solved-for parameters are included such as corrections to the position of the primary and to its pole vector, but these partials are computed at that stage.

From Eq. (6) we obtain the equations for the partial derivatives, the variational equations

$$
\frac{d^2}{dt^2} \left(\frac{\partial c_{3i-2}}{\partial q_k} \right) = \left(\frac{\partial F_{3i-2}}{\partial q_k} \right)_{\text{explicit}} + \sum_{j=1}^{15} \frac{\partial F_{3i-2}}{\partial c_j} \frac{\partial c_j}{\partial q_k}
$$

$$\frac{d^2}{dt^2}\left(\frac{\partial c_{3i-1}}{\partial q_k}\right) = \left(\frac{\partial F_{3i-1}}{\partial q_k}\right)_{\text{explicit}} + \sum_{j=1}^{15} \frac{\partial F_{3i-1}}{\partial c_j}\frac{\partial c_j}{\partial q_k}$$

$$\frac{d^2}{dt^2}\left(\frac{\partial c_{3i}}{\partial q_k}\right) = \left(\frac{\partial F_{3i}}{\partial q_k}\right)_{\text{explicit}} + \sum_{j=1}^{15} \frac{\partial F_{3i}}{\partial c_j}\frac{\partial c_j}{\partial q_k} \tag{11}$$

for $i = 1, 2, \ldots, 5$; $k = 1, 2, \ldots, 39$.

From Eqs. (2),(3),(4) and (7) to (10) we have

$$\begin{aligned}\frac{\partial F_{3i-2}}{\partial c_k} &= \frac{3GM(1+m_i)c_{3i-2}}{r_i^4}\frac{\partial r_i}{\partial c_k} - \frac{GM(1+m_i)\delta_{3i-2,k}}{r_i^3} \\ &+ \sum_{\substack{j=1\\ \neq i}}^{5} GMm_j\Bigg\{\frac{(\delta_{3j-2,k}-\delta_{3i-2,k})}{r_{ij}^3} \\ &- \frac{3(c_{3j-2}-c_{3i-2})}{r_{ij}^4}\frac{\partial r_{ij}}{\partial c_k} \\ &- \frac{\delta_{3j-2,k}}{r_j^3} + \frac{3c_{3j-2}}{r_j^4}\frac{\partial r_j}{\partial c_k}\Bigg\} \\ &+ GM_s\left\{-\frac{\delta_{3i-2,k}}{r_{is}^3} - \frac{3(x_s-c_{3i-2})}{r_{is}^4}\frac{\partial r_{is}}{\partial c_k}\right\} \\ &+ \sum_{p=1}^{2} GM_p\left\{-\frac{\delta_{3i-2,k}}{R_{ip}^3} - \frac{3(x_p-c_{3i-2})}{R_{ip}^4}\frac{\partial R_{ip}}{\partial c_k}\right\} \\ &+ A_i\delta_{3i-2,k} + \frac{\partial A_i}{\partial c_k}c_{3i-2} \\ &+ \sum_{\substack{l=1\\ \neq i}}^{5}\left(m_l A_l \delta_{3l-2,k} + m_l c_{3l-2}\frac{\partial A_l}{\partial c_k}\right)\end{aligned} \tag{12}$$

$$\begin{aligned}\frac{\partial F_{3i-1}}{\partial c_k} &= \frac{3GM(1+m_i)c_{3i-1}}{r_i^4}\frac{\partial r_i}{\partial c_k} - \frac{GM(1+m_i)\delta_{3i-1,k}}{r_i^3} \\ &+ \sum_{\substack{j=1\\ \neq i}}^{5} GMm_j\Bigg\{\frac{(\delta_{3j-1,k}-\delta_{3i-1,k})}{r_{ij}^3} \\ &- \frac{3(c_{3j-1}-c_{3i-1})}{r_{ij}^4}\frac{\partial r_{ij}}{\partial c_k} \\ &- \frac{\delta_{3j-1,k}}{r_j^3} + \frac{3c_{3j-1}}{r_j^4}\frac{\partial r_j}{\partial c_k}\Bigg\} \\ &+ GM_s\left\{-\frac{\delta_{3i-1,k}}{r_{is}^3} - \frac{3(y_s-c_{3i-1})}{r_{is}^4}\frac{\partial r_{is}}{\partial c_k}\right\} \\ &+ \sum_{p=1}^{2} GM_p\left\{-\frac{\delta_{3i-1,k}}{R_{ip}^3} - \frac{3(y_p-c_{3i-1})}{R_{ip}^4}\frac{\partial R_{ip}}{\partial c_k}\right\} \\ &+ A_i\delta_{3i-1,k} + \frac{\partial A_i}{\partial c_k}c_{3i-1}\end{aligned}$$

$$+\sum_{\substack{l=1\\ \neq i}}^{5}\left(m_l A_l \delta_{3l-1,k} + m_l c_{3l-1}\frac{\partial A_l}{\partial c_k}\right) \tag{13}$$

$$\begin{aligned}\frac{\partial F_{3i}}{\partial c_k} = {} & \frac{3GM(1+m_i)c_{3i}}{r_i^4}\frac{\partial r_i}{\partial c_k} - \frac{GM(1+m_i)\delta_{3i,k}}{r_i^3} \\ & + \sum_{\substack{j=1\\ \neq i}}^{5} GMm_j\bigg\{\frac{(\delta_{3j,k}-\delta_{3i,k})}{r_{ij}^3} \\ & - \frac{3(c_{3j}-c_{3i})}{r_{ij}^4}\frac{\partial r_{ij}}{\partial c_k} - \frac{\delta_{3j,k}}{r_j^3} + \frac{3c_{3j}}{r_j^4}\frac{\partial r_j}{\partial c_k}\bigg\} \\ & + GM_s\bigg\{-\frac{\delta_{3i,k}}{r_{is}^3} - \frac{3(z_s - c_{3i})}{r_{is}^4}\frac{\partial r_{is}}{\partial c_k}\bigg\} \\ & + \sum_{p=1}^{2} GM_p\bigg\{-\frac{\delta_{3i,k}}{R_{ip}^3} - \frac{3(z_p - c_{3i})}{R_{ip}^4}\frac{\partial R_{ip}}{\partial c_k}\bigg\} \\ & + A_i\delta_{3i,k} + \frac{\partial A_i}{\partial c_k}c_{3i} + \frac{\partial B_i}{\partial c_k} \\ & + \sum_{\substack{l=1\\ \neq i}}^{5}\left(m_l A_l \delta_{3l,k} + m_l c_{3l}\frac{\partial A_l}{\partial c_k} + m_l\frac{\partial B_l}{\partial c_k}\right)\end{aligned} \tag{14}$$

where

$$\begin{aligned}\frac{\partial A_i}{\partial c_k} = {} & -\frac{GM}{r_i^4}\frac{\partial r_i}{\partial c_k}\sum_{n=2}^{4}\frac{J_n R_u^n P''_{n+2}}{r_i^n} \\ & + \frac{GM\delta_{3i,k}}{r_i^4}\sum_{n=2}^{4}\frac{J_n R_u^n P''_{n+1}}{r_i^n}\end{aligned} \tag{15}$$

$$\begin{aligned}\frac{\partial B_i}{\partial c_k} = {} & \frac{GM}{r_i^3}\frac{\partial r_i}{\partial c_k}\sum_{n=2}^{4}\frac{J_n R_u^n P''_{n+1}}{r_i^n} \\ & - \frac{GM\delta_{3i,k}}{r_i^3}\sum_{n=2}^{4}\frac{J_n R_u^n P''_{n}}{r_i^n}\end{aligned} \tag{16}$$

with P'_n and P''_n obtained from the recurrence relations

$$\begin{aligned}(n+1)P_{n+1} &= (2n+1)P_n\cdot(c_{3i}/r_i) - nP_{n-1} \\ P'_{n+1} &= (n+1)P_n + P'_n\cdot(c_{3i}/r_i) \\ P''_{n+1} &= (n+2)P'_n + P''_n\cdot(c_{3i}/r_i)\end{aligned} \tag{17}$$

with $P_0 = 1, P_1 = c_{3i}/r_i$

and

$$
\begin{aligned}
\frac{\partial r_i}{\partial c_k} &= \frac{1}{r_i}(c_{3i-2}\delta_{3i-2,k} + c_{3i-1}\delta_{3i-1,k} + c_{3i}\delta_{3i,k}) \\
\frac{\partial r_{ij}}{\partial c_k} &= \frac{1}{r_{ij}}\Big\{(c_{3j-2} - c_{3i-2})(\delta_{3j-2,k} - \delta_{3i-2,k}) \\
&\quad + (c_{3j-1} - c_{3i-1})(\delta_{3j-1,k} - \delta_{3i-1,k}) \\
&\quad + (c_{3j} - c_{3i})(\delta_{3j,k} - \delta_{3i,k})\Big\} \\
\frac{\partial r_{is}}{\partial c_k} &= -\frac{1}{r_{is}}\Big\{(x_s - c_{3i-2})\delta_{3i-2,k} \\
&\quad + (y_s - c_{3i-1})\delta_{3i-1,k} + (z_s - c_{3i})\delta_{3i,k}\Big\} \\
\frac{\partial R_{ip}}{\partial c_k} &= -\frac{1}{R_{ip}}\Big\{(x_p - c_{3i-2})\delta_{3i-2,k} \\
&\quad + (y_p - c_{3i-1})\delta_{3i-1,k} + (z_p - c_{3i})\delta_{3i,k}\Big\}
\end{aligned}
\tag{18}
$$

and δ_{ij} is the Kronecker delta.

Some simplification in programming the partials follows from their symmetry. From Eqs. (12) to (18) it can be shown

$$
\frac{\partial F_i}{\partial c_j} = \frac{\partial F_j}{\partial c_i}
\tag{19}
$$

for each set of $i, j = (3k-2, 3k-1, 3k), k = 1, 2, \ldots, 5$.

In Eq. (11) the explicit partial derivatives of F_{3i-2}, F_{3i-1} and F_{3i} with respect to q_k are zero for $k = 1, 2, \ldots, 30$, since the F_{3i-2}, F_{3i-1} and F_{3i} do not depend on the initial position and velocities of the satellites. The remaining explicit partial derivatives are with respect to the oblateness parameters and masses. Reverting to using these symbols we have

$$
\begin{aligned}
\frac{\partial F_{3i-2}}{\partial J_j} &= \frac{\partial A_i}{\partial J_j} c_{3i-2} + \sum_{\substack{l=1 \\ \neq i}}^{5} m_l \frac{\partial A_l}{\partial J_j} c_{3l-2} \quad (j = 2, 3, 4) \\
\frac{\partial F_{3i-2}}{\partial m_i} &= -\frac{GMc_{3i-2}}{r_i^3}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial F_{3i-2}}{\partial m_j} &= GM\left(\frac{c_{3j-2} - c_{3i-2}}{r_{ij}^3} - \frac{c_{3j-2}}{r_j^3}\right) \\
&\quad + A_j c_{3j-2} \quad (i \neq j) \\
\frac{\partial F_{3i-2}}{\partial M} &= -\frac{G(1+m_i)c_{3i-2}}{r_i^3}
\end{aligned}
$$

$$
+ \sum_{\substack{j=1 \\ \neq i}}^{5} G m_j \left(\frac{c_{3j-2} - c_{3i-2}}{r_{ij}^3} - \frac{c_{3j-2}}{r_j^3} \right)
$$

$$
+ \frac{A_i c_{3i-2}}{M} + \sum_{\substack{l=1 \\ \neq i}}^{5} \frac{m_l A_l c_{3l-2}}{M} \tag{20}
$$

$$
\begin{aligned}
\frac{\partial F_{3i-1}}{\partial J_j} &= \frac{\partial A_i}{\partial J_j} c_{3i-1} + \sum_{\substack{l=1 \\ \neq i}}^{5} m_l \frac{\partial A_l}{\partial J_j} c_{3l-1} \quad (j = 2, 3, 4) \\
\frac{\partial F_{3i-1}}{\partial m_i} &= -\frac{G M c_{3i-1}}{r_i^3} \\
\frac{\partial F_{3i-1}}{\partial m_j} &= G M \left(\frac{c_{3j-1} - c_{3i-1}}{r_{ij}^3} - \frac{c_{3j-1}}{r_j^3} \right) \\
&\quad + A_j c_{3j-1} \quad (i \neq j) \\
\frac{\partial F_{3i-1}}{\partial M} &= -\frac{G(1 + m_i) c_{3i-1}}{r_i^3} \\
&\quad + \sum_{\substack{j=1 \\ \neq i}}^{5} G m_j \left(\frac{c_{3j-1} - c_{3i-1}}{r_{ij}^3} - \frac{c_{3j-1}}{r_j^3} \right) \\
&\quad + \frac{A_i c_{3i-1}}{M} + \sum_{\substack{l=1 \\ \neq i}}^{5} \frac{m_l A_l c_{3l-1}}{M}
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
\frac{\partial F_{3i}}{\partial J_j} &= \frac{\partial A_i}{\partial J_j} c_{3i} + \frac{\partial B_i}{\partial J_j} \\
&\quad + \sum_{\substack{l=1 \\ \neq i}}^{5} \left(m_l \frac{\partial A_l}{\partial J_j} c_{3l} + m_l \frac{\partial B_l}{\partial J_j} \right) \quad (j = 2, 3, 4) \\
\frac{\partial F_{3i}}{\partial m_i} &= -\frac{G M c_{3i}}{r_i^3} \\
\frac{\partial F_{3i}}{\partial m_j} &= G M \left(\frac{c_{3j} - c_{3i}}{r_{ij}^3} - \frac{c_{3j}}{r_j^3} \right) \\
&\quad + A_j c_{3j} + B_j \quad (i \neq j)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial F_{3i}}{\partial M} &= -\frac{G(1 + m_i) c_{3i}}{r_i^3} \\
&\quad + \sum_{\substack{j=1 \\ \neq i}}^{5} G m_j \left(\frac{c_{3j} - c_{3i}}{r_{ij}^3} - \frac{c_{3j}}{r_j^3} \right) \\
&\quad + \frac{A_i c_{3i}}{M} + \frac{B_i}{M} + \sum_{\substack{l=1 \\ \neq i}}^{5} \left(\frac{m_l A_l c_{3l}}{M} + \frac{m_l B_l}{M} \right)
\end{aligned} \tag{22}
$$

where

$$\frac{\partial A_i}{\partial J_j} = \frac{GMR_u^j P'_{j+1}}{r_i^{3+j}}$$
$$\frac{\partial B_i}{\partial J_j} = -\frac{GMR_u^j P'_j}{r_i^{2+j}} \quad (23)$$

for $j = 2, 3, 4$.

3. Approximate orbits

3.1 Theories

Orbital models based on a precessing ellipse were fitted to the differenced catalogue of observations described in Taylor (1998) and Voyager data. Although these orbits are inadequate to analyse the modern data they are still useful as a quick and easy means to compute residuals and test the quality of a series of observations. They also provide a way to generate approximate starting conditions for the numerical integration.

Ariel, Umbriel, Titania and Oberon have small eccentricities and inclinations to the equatorial plane. Miranda also has a small eccentricity but an inclination of $\approx 4^\circ\!.3$ to the equator. For small eccentricities and inclinations the pericentres and nodes are difficult to determine. For Ariel, Umbriel, Titania and Oberon it is better to use in place of e, the eccentricity and ϖ, the longitude of pericentre, $h = e \sin \varpi$, $k = e \cos \varpi$, and in place of i, the inclination and Ω, the longitude of the node $p = \sin i \sin \Omega$ and $q = \sin i \cos \Omega$. For Miranda we use h and k but retain the elements i and Ω.

The general form for the theories of Ariel, Umbriel, Titania and Oberon is

$$a = a_0$$
$$\lambda = \lambda_0 + nt$$
$$h = h_0 \cos \beta t + k_0 \sin \beta t$$
$$k = k_0 \cos \beta t - h_0 \sin \beta t$$
$$p = p_0 \cos \gamma t + q_0 \sin \gamma t$$
$$q = q_0 \cos \gamma t - p_0 \sin \gamma t \quad (24)$$

where a is the semi-major axis, λ is the mean longitude, n is the mean motion and β, γ are the rates of change of the apse and node respectively. a_0, λ_0,h_0,k_0,p_0 and q_0 are constants determined from the fit to observations. For Miranda the same orbital model is used but p, q are replaced by $i = i_0$, $\Omega = \Omega_0 + \dot{\Omega} t$, where i_0, Ω_0 are obtained from the fit to data. The longitudes are measured on the Uranus mean equator of 1950.0 from the intersection of this plane with the Earth mean equator of 1950.0.

Let the subscripts i = 1,2,3 refer to satellites Miranda, Ariel and Umbriel respectively. To their mean longitudes we add the periodic terms

$$\Delta L_i = A_i(c'_i \sin(\theta_0 + \dot{\theta} t) + c''_i \sin 2(\theta_0 + \dot{\theta} t)). \quad (25)$$

These perturbations arise from the Laplacian resonance $n_1 - 3n_2 + 2n_3 = -0^\circ\!.0785/d$ and is the largest perturbation in the satellite system (Laskar and Jacobson 1987). The coefficients c'_i ,c''_i are taken from Lazzaro et al. (1984) and the amplitudes A_i and phase and frequency of the resonance obtained from the fit to data.

3.2 **Expansion of the satellite vector**

In Fig.1 the various reference planes needed to define the satellite theory are shown with the satellite at position P in its orbit.

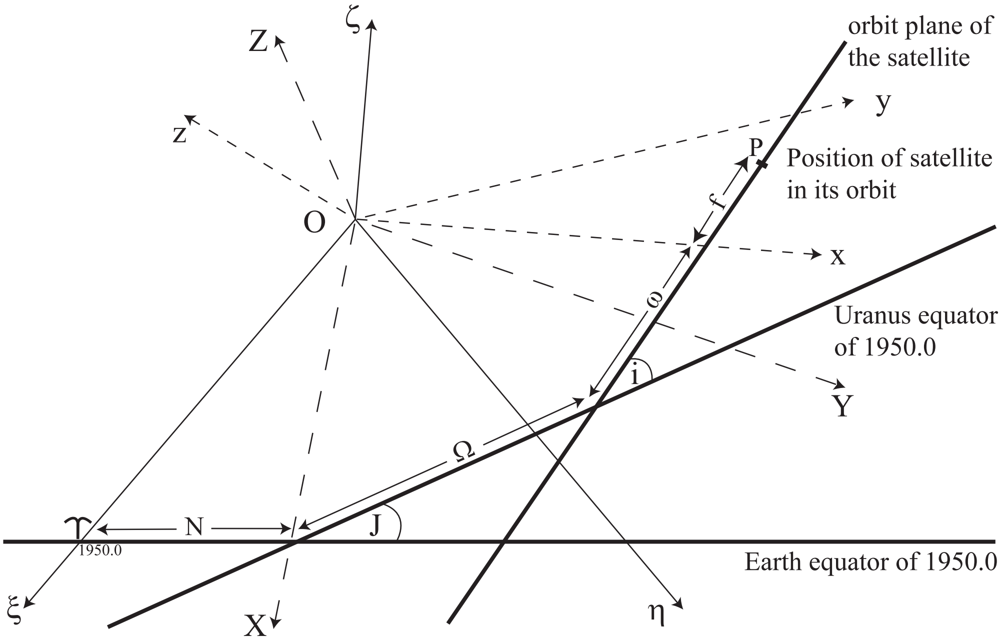

Fig. 1 Reference systems for defining the Uranian satellite theories

From Fig.1 referring the satellite vector OP in the reference system O_{xyz}, where O_x is the axis from the centre of Uranus to the pericentre of the satellite and O_y is in the plane of the satellite orbit, to the system O_{XYZ}, where O_X is the intersection of the mean equator of Uranus and the Earth for 1950.0 and O_Y in the plane of the equator of Uranus we have

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = ABC \begin{pmatrix} r\cos f \\ r\sin f \\ 0 \end{pmatrix} \tag{26}$$

where

$$A = \begin{pmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{27}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{pmatrix} \tag{28}$$

$$C = \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{29}$$

From expressions

$$r = a(1 + \frac{e^2}{2} - e\cos l - \frac{e^2}{2}\cos 2l + O(e^3)) \tag{30}$$

and

$$f = l + 2e\sin l + \frac{5}{4}e^2\sin 2l + O(e^3) \tag{31}$$

the quantities $r\cos f$ and $r\sin f$ up to and including terms of order e^2 are

$$r\cos f = a\Big\{-\frac{3}{2}e + (1-\frac{3}{8}e^2)\cos l + \frac{e}{2}\cos 2l + \frac{3}{8}e^2\cos 3l\Big\} \tag{32}$$

$$r\sin f = a\Big\{(1-\frac{5}{8}e^2)\sin l + \frac{e}{2}\sin 2l + \frac{3}{8}e^2\sin 3l\Big\}. \tag{33}$$

These could also be obtained directly from Cayley's Tables (1861). Substituting $r\cos f$ and $r\sin f$ from Eqs.(32) and (33) into Eq.(26) and ignoring powers of $\sin i$ greater than the second we find from Eqs.(26) to (29) the satellite vector for Ariel, Umbriel, Titania and Oberon in terms of a, λ, h, k, p and q is

$$\begin{aligned} X = a\Big\{&-\frac{3k}{2} + \left(1-\frac{3}{8}k^2-\frac{5}{8}h^2-\frac{1}{2}p^2\right)\cos\lambda \\ &+\left(\frac{1}{4}hk+\frac{1}{2}pq\right)\sin\lambda + \frac{1}{2}k\cos 2\lambda + \frac{1}{2}h\sin 2\lambda \\ &+\frac{3}{8}(k^2-h^2)\cos 3\lambda + \frac{3}{4}hk\sin 3\lambda\Big\} \end{aligned} \tag{34}$$

$$\begin{aligned} Y = a\Big\{&-\frac{3}{2}h + \left(\frac{1}{4}hk+\frac{1}{2}pq\right)\cos\lambda \\ &+\left(1-\frac{5}{8}k^2-\frac{3}{8}h^2-\frac{1}{2}q^2\right)\sin\lambda - \frac{1}{2}h\cos 2\lambda \\ &+\frac{1}{2}k\sin 2\lambda - \frac{3}{4}hk\cos 3\lambda + \frac{3}{8}(k^2-h^2)\sin 3\lambda\Big\} \end{aligned} \tag{35}$$

$$\begin{aligned} Z = a\Big\{&-\frac{3}{2}hq + \frac{3}{2}kp - p\cos\lambda + q\sin\lambda \\ &-\frac{1}{2}(hq+kp)\cos 2\lambda + \frac{1}{2}(kq-hp)\sin 2\lambda\Big\}. \end{aligned} \tag{36}$$

For Miranda we again substitute Eqs.(32) and (33) into Eq.(26) but do not expand in terms of $\sin i$. We find from Eqs.(26) to (29) the satellite vector in terms of a, λ, h, k, i and Ω is

$$X = X'\cos\Omega - Y'\sin\Omega\cos i \tag{37}$$

$$Y = X'\sin\Omega + Y'\cos\Omega\cos i \tag{38}$$

$$Z = Y'\sin i \tag{39}$$

where

$$\begin{aligned} X' = a\Big\{&-\frac{3}{2}k\cos\Omega - \frac{3}{2}h\sin\Omega \\ &+\left(1-\frac{k^2}{2}-\frac{h^2}{2}\right)\cos(\lambda-\Omega) \\ &-\frac{1}{8}(h^2-k^2)\cos(\lambda+\Omega) + \frac{kh}{4}\sin(\lambda+\Omega) \\ &+\frac{1}{2}k\cos(2\lambda-\Omega) + \frac{1}{2}h\sin(2\lambda-\Omega) \\ &+\frac{3}{8}(k^2-h^2)\cos(3\lambda-\Omega) + \frac{3}{4}hk\sin(3\lambda-\Omega)\Big\} \end{aligned} \tag{40}$$

$$
\begin{aligned}
Y' = a\Big\{ & -\frac{3}{2}h\cos\Omega + \frac{3}{2}k\sin\Omega \\
& + \left(1 - \frac{h^2}{2} - \frac{k^2}{2}\right)\sin(\lambda - \Omega) \\
& + \frac{hk}{4}\cos(\lambda + \Omega) + \frac{1}{8}(h^2 - k^2)\sin(\lambda + \Omega) \\
& - \frac{1}{2}h\cos(2\lambda - \Omega) + \frac{1}{2}k\sin(2\lambda - \Omega) \\
& - \frac{3}{4}hk\cos(3\lambda - \Omega) + \frac{3}{8}(k^2 - h^2)\sin(3\lambda - \Omega)\Big\}.
\end{aligned}
\tag{41}
$$

Let (ξ, η, ζ) be the components of (X, Y, Z) in a system in which the O_ξ axis points towards the equinox of 1950.0, and O_η is in the plane of the Earth's equator for 1950.0 (see Fig. 1). So we have

$$
\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} \cos N & -\sin N\cos J & \sin N\sin J \\ \sin N & \cos N\cos J & -\cos N\sin J \\ 0 & \sin J & \cos J \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}
\tag{42}
$$

where J and N are the inclination and longitude of the node of the Uranus equator of 1950.0 on the Earth equator of 1950.0. The values $J = 74\overset{\circ}{.}96$ and $N = 166\overset{\circ}{.}72$ were used, taken from Dunham (1971). In the fit of the numerical integration to observations the more recent values $J = 74\overset{\circ}{.}8883$ and $N = 166\overset{\circ}{.}5969$ were used. These were derived from the α, δ for the pole of the equatorial plane of Uranus in French et al (1988) noting $J = 90^\circ - \delta$, $N = 90^\circ + \alpha$.

3.3 **Fit to observations**

The theories were fitted to the differenced catalogue and Voyager data by least-squares. This is described in Taylor (1998). For each satellite, corrections were sought to a_0, λ_0, h_0, k_0, p_0 and q_0 (i_0, Ω_0 replacing p_0, q_0 for Miranda), n, β and γ ($\dot{\Omega}$ replacing γ for Miranda) and in addition corrections to the parameters A_1, A_2, A_3, θ_0 and $\dot{\theta}$ which model the Laplacian term. Initial values for the parameters were obtained from the solution in Veillet (1983) which has as epoch 2433282.0 with small non-zero values taken for the amplitudes A_2 and A_3. The solution found is given in Table 1 together with the formal errors from the least-squares for each solved for parameter. The units are degrees/day for the mean motions, secular rates and frequency of the Laplacian resonance term, degrees for the longitudes and phase of the Laplacian resonance term and AU's for the semi-major axes. In addition to the parameters solved-for in Table 1, corrections to the position of Uranus from DE200 were found. These were obtained using meridian circle observations from La Palma made in the period 1992 to 1994. The corrections found were in RA $-0\overset{''}{.}231 \pm 0\overset{''}{.}055$ and in Dec $-0\overset{''}{.}145 \pm 0\overset{''}{.}043$.

Table 1 Parameters in the theories

	Miranda		Ariel
a_0	$0.86799316439842D-03 \pm 0.25819046D-07$	a_0	$0.12761445775466D-02 \pm 0.55221683D-07$
λ_0	$0.11326978648290D+03 \pm 0.65388107D-01$	λ_0	$0.15582451708408D+03 \pm 0.23556829D-01$
h_0	$0.10192937014337D-02 \pm 0.49373438D-03$	h_0	$0.18147114988730D-02 \pm 0.42247103D-04$
k_0	$-0.10635161958324D-02 \pm 0.47306164D-03$	k_0	$-0.27972048760168D-03 \pm 0.25098554D-03$
i_0	$0.43115927221656D+01 \pm 0.90613791D-02$	p_0	$-0.15363209334399D-02 \pm 0.66260450D-03$
Ω_0	$0.36367139947984D+02 \pm 0.21802516D+01$	q_0	$0.13771382587122D-02 \pm 0.75394867D-03$
n	$0.25469066544926D+03 \pm 0.47681751D-05$	n	$0.14283565402133D+03 \pm 0.17051777D-05$
β	$0.42057175083412D-01 \pm 0.20183384D-02$	β	$0.17966181323277D-01 \pm 0.60271316D-03$
$\dot{\Omega}$	$-0.55738385882661D-01 \pm 0.16521371D-03$	γ	$0.15821522602431D+00 \pm 0.20888063D-02$
	Umbriel		**Titania**
a_0	$0.17781404528507D-02 \pm 0.77942848D-07$	a_0	$0.29165594007616D-02 \pm 0.11391535D-06$
λ_0	$0.28518029718670D+03 \pm 0.17094128D-01$	λ_0	$0.90900980895531D+00 \pm 0.56446064D-02$
h_0	$-0.14546329622904D-02 \pm 0.15789071D-03$	h_0	$0.12866515239454D-02 \pm 0.53175645D-04$
k_0	$-0.38537513621134D-02 \pm 0.62779697D-04$	k_0	$-0.20178254937957D-02 \pm 0.38729815D-04$
p_0	$-0.33810680501144D-02 \pm 0.31943836D-03$	p_0	$-0.71273377813200D-03 \pm 0.23229810D-03$
q_0	$-0.14640055025941D-02 \pm 0.65164723D-03$	q_0	$0.33370331371709D-02 \pm 0.14566644D-03$
n	$0.86868879962177D+02 \pm 0.12585240D-05$	n	$0.41351417741106D+02 \pm 0.44432643D-06$
β	$0.80024386762851D-02 \pm 0.17860349D-03$	β	$0.50514522672439D-02 \pm 0.11298237D-03$
γ	$0.31243322013085D-03 \pm 0.84153869D-03$	γ	$-0.27588010015215D-02 \pm 0.31149640D-03$
	Oberon		**Laplacian resonance parameters**
a_0	$0.38993657406356D-02 \pm 0.70792496D-07$	A_1	$-0.12146031188447D+00 \pm 0.85637041D-03$
λ_0	$0.16963839746138D+03 \pm 0.41413785D-02$	A_2	$-0.16733405675459D-02 \pm 0.11897354D-03$
h_0	$-0.69246930843389D-03 \pm 0.19639850D-04$	A_3	$-0.78412576161838D-03 \pm 0.93956694D-04$
k_0	$0.22169620928800D-03 \pm 0.44910847D-04$	θ_0	$0.22035206082295D+03 \pm 0.56471259D+01$
p_0	$-0.96378687945889D-03 \pm 0.12403533D-03$	$\dot{\theta}$	$-0.78890334827480D-01 \pm 0.43102383D-03$
q_0	$-0.14802532902556D-02 \pm 0.91549265D-04$		
n	$0.26739483403119D+02 \pm 0.31627924D-06$		
β	$-0.78114170909297D-02 \pm 0.27989117D-03$		
γ	$0.34255341945565D-01 \pm 0.34245721D-03$		

The statistics obtained for the different datasets analysed is left for presentation in an account which discusses each dataset in the catalogue in detail. In Table 2 we give the accuracy of some of the best quality observations in the catalogue made during the entire observing period of the satellites. In the table σ is the standard deviation about the mean and the type of observations made are: V = visual estimations, M = micrometer, P = photographic and C = CCD. The table clearly shows how the accuracy of the observations greatly improves when improvements in the observing and reduction techniques are made.

Dataset	Year(s) of observations	σ ($''$)	Type of observations
Herschel (1815)	1787 - 98	4.33	V,M
Herschel (1833)	1787 - 98 ; 1828 - 32	2.20	M
Lamont (1837)	1837	1.46	M
Lassell (1853)	1852 - 53	0.84	M
Lassell (1865)	1863 - 65	0.49	M
Newcomb (1875)	1874 - 75	0.30	M
USNO (1883)	1883	0.29	M
Schaeberle (1897)	1895, 97	0.28	M
Aitken (1909)	1906 - 07	0.26	M
Harris (1949)[1]	1914 - 16	0.16	P
Struve (1928)	1927 - 28	0.32	M
van Biesbroeck et al (1976)[2]	1966	0.14	P
Veillet (1983)[3]	1980 - 82	0.07	P
Jones et al (1998)	1990 - 91	0.04	C

Table 2 Statistics of selected datasets

1 observations made at Lowell Obs.
2 observations made at the Catalina Stn. of the Lunar and Planetary Laboratory
3 observations made at ESO

4. **Determination of starting conditions for the numerical integration**

4.1 **Initial estimates**

To get an initial estimate of the positions and velocities of the integration, it was first fitted to the analytical theories described in Sect.3. The epoch chosen was Jan. 5.0 1987 (JED 2446800.5) with positions and velocities for the satellites at this epoch obtained from elements from the theories referred to the equatorial plane of Uranus and then using the transformation formulae in Appendix A. Positions from the analytical theories were output in the reference plane of the integration. Then the initial states were determined from a least-squares fit to these positions over firstly a short time interval ($\pm$10d from epoch with positions at 0.5d was used) and then to longer intervals until a set of initial positions and velocities are found from a fit to the time interval covering the period we want to fit to the data. These are then used as initial estimates to fit the numerical integration to the observations.

4.2 **Determination of osculating elements at epoch**

Using the initial estimates from the fit of the numerical integration to the theories as starting conditions for a fit of the numerical integration to the observations it was found from successive iterations a large number of observations were rejected. This indicated the initial estimates were not close enough to the solution. To improve these estimates the classical osculating elements at epoch were solved for in place of position and velocity.

The numerical integration of the equations of motion and variational equations was still performed in rectangular coordinates. The initial position and velocity for each satellite at epoch were calculated from the current estimate of the initial osculating elements using two-body formulae (see Appendix A). In the equations of condition the partial derivatives with respect to initial position and velocity $\frac{\partial c_i}{\partial q_k}, i = 1, 2, \ldots, 15$; $k = 1, 2, \ldots, 30$ are converted to partial derivatives with respect to the initial osculating elements using the formulae for $i = 1, 2, \ldots, 5$

$$\frac{\partial c_{3i-2}}{\partial p_k} = \sum_{j=3i-2}^{3i} \left(\frac{\partial c_{3i-2}}{\partial q_j} \frac{\partial q_j}{\partial p_k} + \frac{\partial c_{3i-2}}{\partial q_{15+j}} \frac{\partial q_{15+j}}{\partial p_k} \right) \tag{43}$$

$$\frac{\partial c_{3i-1}}{\partial p_k} = \sum_{j=3i-2}^{3i} \left(\frac{\partial c_{3i-1}}{\partial q_j} \frac{\partial q_j}{\partial p_k} + \frac{\partial c_{3i-1}}{\partial q_{15+j}} \frac{\partial q_{15+j}}{\partial p_k} \right) \tag{44}$$

$$\frac{\partial c_{3i}}{\partial p_k} = \sum_{j=3i-2}^{3i} \left(\frac{\partial c_{3i}}{\partial q_j} \frac{\partial q_j}{\partial p_k} + \frac{\partial c_{3i}}{\partial q_{15+j}} \frac{\partial q_{15+j}}{\partial p_k} \right) \tag{45}$$

$$, k = 6i - 5, \ldots, 6i$$

where $p_{6i-5}, p_{6i-4}, \ldots, p_{6i}, i = 1, 2, \ldots, 5$ are the initial osculating elements for Miranda, Ariel, Umbriel, Titania and Oberon respectively. The partial derivatives $\frac{\partial q_j}{\partial p_k}$ in Eqs. (43),(44) and (45) are calculated from the formulae for elliptic motion (see Appendix C).

Corrections to the osculating elements $p_k, k = 1, \ldots, 30$ are then found from the fit of the integration to the observations, using least-squares. The corrected elements are then converted to position and velocity by formulae of elliptic motion and these are used as starting values for a new integration of the equations of motion and variational equations. After a few iterations a converged solution was attained. From this solution and reverting to the partial derivatives with respect to the initial position and velocity at epoch a solution for the initial state vectors was found. The solutions discussed in Taylor (1998) were obtained using the rectangular coordinates formulation.

In the determination of the starting conditions all other parameters were kept fixed at previously published values. Some further solutions were tried iterating on osculating elements at epoch together with physical parameters of the system. A solution was attempted for the osculating elements, the masses of Miranda, Ariel, Titania and Oberon, the mass of Uranus, J_2 and corrections to the position of Uranus. This failed to converge revealing significant correlations between the semi-major axes of the satellites with the satellite masses e.g. a correlation coefficient of 0·95 between the semi-major axis of Titania and the mass of Oberon, 0·93 between the semi-major axis of Oberon and the mass of Titania and also between the semi-major axes of the satellites and J_2 e.g. a correlation coefficient of 0·95 between the semi-major axis of Miranda and J_2. The reason why convergence of solutions in coordinates and velocity components with physical parameters is possible as described in Taylor (1998) but not in osculating elements with physical parameters is not at present entirely understood. It is the subject of further study.

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Appendix A

To compute the position and velocity of a satellite from the osculating elements, the mass of the primary and the mass of the satellite.

Let the osculating elements of the satellite be

$$\begin{aligned} a &= \text{semi-major axis} \\ e &= \text{eccentricity} \\ i &= \text{inclination} \\ \lambda &= \text{mean longitude} \\ \varpi &= \text{longitude of the apse} \\ \Omega &= \text{longitude of the node} \end{aligned}$$

and

$$\begin{aligned} M_p &= \text{mass of primary (unit is mass of Sun)} \\ m &= \text{mass of satellite (unit is mass of primary).} \end{aligned}$$

We have

$$\begin{aligned} \mu &= k^2(M_p + mM_p) \\ &= k^2 M_p(1+m) \end{aligned} \qquad (A.1)$$

where k is the Gaussian gravitational constant.

Let n be the mean motion of the satellite then from Kepler's 3rd law

$$\mu = n^2 a^3 \qquad (A.2)$$

we have from Eqs. (A.1) and (A.2)

$$n = k\left(\frac{M_p(1+m)}{a^3}\right)^{\frac{1}{2}}. \qquad (A.3)$$

If n is in radians/day then k=0.01720209895.

We have Kepler's equation

$$E - e\sin E = l \qquad (A.4)$$

where

$$\begin{aligned} E &= \text{eccentric anomaly} \\ l &= \text{mean anomaly } (\lambda - \varpi)\cdot \end{aligned}$$

Kepler's equation is solved by the Newton-Raphson method

$$E_{i+1} = E_i - \frac{(E_i - e \sin E_i - l)}{(1 - e \cos E_i)}$$

with the initial value $E_0 = l + e \sin l$.

Given an accuracy tolerance ϵ the iterative procedure is terminated when $|E_{i+1} - E_i| < \epsilon$ where E_{i+1}, E_i are successive approximations.

Let E be the value for the eccentric anomaly corresponding to the given value of l and e which together satisfy Eq. (A.4).

The satellite coordinates then are obtained from

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = ABC \begin{pmatrix} a(\cos E - e) \\ a\sqrt{1 - e^2} \sin E \\ 0 \end{pmatrix} \tag{A.5}$$

where

$$A = \begin{pmatrix} \cos \Omega & -\sin \Omega & 0 \\ \sin \Omega & \cos \Omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{A.6}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{pmatrix} \tag{A.7}$$

$$C = \begin{pmatrix} \cos \omega & -\sin \omega & 0 \\ \sin \omega & \cos \omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{A.8}$$

$\omega = \varpi - \Omega$ is the argument of pericentre.

Now as we have osculating elements, which define an instantaneous Keplerian ellipse, only l is a function of time. Hence as E is connected to l, from Eq.(A.4) we find differentiating Eq. (A.5) with respect to time

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = ABC \begin{pmatrix} -a \sin E \dot{E} \\ a\sqrt{1 - e^2} \cos E \dot{E} \\ 0 \end{pmatrix} \tag{A.9}$$

with A,B,C from Eqs.(A.6),(A.7) and (A.8) respectively. $\dot{E}$ is found from differentiation of Eq.(A.4) and noting $r = a(1 - e \cos E)$, the radius vector. We have

$$\dot{E} = \frac{na}{r}. \tag{A.10}$$

Hence the coordinates and velocity components are calculated and they will be referred to the reference plane and axes in which the osculating elements are defined.

Appendix B

To compute the osculating elements of a satellite from the position and velocity, the mass of the primary and the mass of the satellite.

Let x,y,z be the coordinates and $\dot{x}$,$\dot{y}$,$\dot{z}$ the velocity components of the satellite.

Suppose also

$$\begin{aligned} M_p &= \text{ mass of primary (unit is mass of Sun)} \\ m &= \text{ mass of satellite (unit is mass of primary).} \end{aligned}$$

We have

$$\begin{aligned} \mu &= k^2(M_p + mM_p) \\ &= k^2 M_p(1+m) \end{aligned} \tag{B.1}$$

where k is the Gaussian gravitational constant.

Now from the coordinates and velocity components we have

$$r = (x^2 + y^2 + z^2)^{\frac{1}{2}} \tag{B.2}$$

$$\dot{r} = (x\dot{x} + y\dot{y} + z\dot{z})/r \tag{B.3}$$

$$v = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{\frac{1}{2}}. \tag{B.4}$$

From Eqs.(B.1),(B.2) and (B.4) the semi-major axis a is computed from

$$a = 1/((2/r) - (v^2/\mu)). \tag{B.5}$$

From

$$r = a(1 - e\cos E) \tag{B.6}$$

and its derivative with respect to time

$$\dot{r} = ae\sin E\dot{E} \tag{B.7}$$

where e is the eccentricity, E is the eccentric anomaly and $\dot{E}$ the derivative of the eccentric anomaly with respect to time

$$e\cos E = 1 - \frac{r}{a} \tag{B.8}$$

and

$$e\sin E = \frac{\dot{r}}{a\dot{E}}. \tag{B.9}$$

From Kepler's equation

$$E - e\sin E = l \tag{B.10}$$

where l is the mean anomaly we find on differentiation with respect to time and using Eq.(B.6)

$$\dot{E} = \frac{na}{r} \tag{B.11}$$

where n is the mean motion of the satellite.

From Kepler's 3rd law

$$n = (\mu/a^3)^{\frac{1}{2}}. \tag{B.12}$$

If n is in radians/day then $k = 0{\cdot}01720209895$. Substituting for n from Eq.(B.12) into Eq.(B.11) we have

$$\dot{E} = \frac{1}{r}\sqrt{\frac{\mu}{a}}. \tag{B.13}$$

From Eq.(B.8) and Eq.(B.9) after substituting for $\dot{E}$ from Eq.(B.13) the eccentricity and eccentric anomaly are computed from

$$e = \left(\left(1 - \frac{r}{a}\right)^2 + \frac{\dot{r}^2 r^2}{a\mu}\right)^{\frac{1}{2}} \tag{B.14}$$

$$E = \tan^{-1}\left((r\dot{r}/\sqrt{\mu a})/(1 - (r/a))\right). \tag{B.15}$$

The mean anomaly l can now be computed from Eq.(B.10) using values of e and E from Eqs.(B.14) and (B.15).

The normal unit vector to the orbit plane of the satellite is

$$(\sin i \sin \Omega, -\sin i \cos \Omega, \cos i)$$

where i is the inclination of the orbit plane of the satellite to the reference plane and Ω is the longitude of the node.

Now by definition the angular momentum vector $\mathbf{h}$ where

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} \tag{B.16}$$

is also perpendicular to the orbit plane. We can thus write this vector as

$$(h \sin i \sin \Omega, -h \sin i \cos \Omega, h \cos i) \tag{B.17}$$

where h is the magnitude of this vector.

Now

$$\mathbf{r} \times \mathbf{v} = (y\dot{z} - z\dot{y}, z\dot{x} - x\dot{z}, x\dot{y} - y\dot{x}). \tag{B.18}$$

Let

$$h_x = y\dot{z} - z\dot{y}, \;\; h_y = z\dot{x} - x\dot{z}, \;\; h_z = x\dot{y} - y\dot{x} \qquad (B.19)$$

and so h_x, h_y and h_z are known.

Then from Eq.(B.17)

$$h_x = h \sin i \sin \Omega, \;\; h_y = -h \sin i \cos \Omega, \;\; h_z = h \cos i, \qquad (B.20)$$

where

$$h = ({h_x}^2 + {h_y}^2 + {h_z}^2)^{\frac{1}{2}}. \qquad (B.21)$$

From the first two formulae in Eq.(B.20)

$${h_x}^2 + {h_y}^2 = h^2 \sin^2 i$$

and so

$$\sin i = \left(\frac{{h_x}^2 + {h_y}^2}{h^2} \right)^{\frac{1}{2}}. \qquad (B.22)$$

From Eqs.(B.20) and (B.22)

$$\begin{aligned} i &= \tan^{-1}\left(\frac{\sin i}{\cos i} \right) \\ &= \tan^{-1}\Big((({h_x}^2 + {h_y}^2)/h^2)^{\frac{1}{2}} / (h_z/h) \Big). \qquad (B.23) \end{aligned}$$

Again from the first two formulae in Eq.(B.20) we have

$$\begin{aligned} \sin \Omega &= h_x / h \sin i \\ \cos \Omega &= -h_y / h \sin i \end{aligned}$$

and so

$$\begin{aligned} \Omega &= \tan^{-1}\left(\frac{\sin \Omega}{\cos \Omega} \right) \\ &= \tan^{-1}\Big((h_x/h \sin i)/(-h_y/h \sin i) \Big). \qquad (B.24) \end{aligned}$$

From the formulae of elliptic motion we have

$$a(\cos E - e) = r \cos f \qquad (B.25)$$

$$a\sqrt{1 - e^2} \sin E = r \sin f \qquad (B.26)$$

where f is the true anomaly. On substituting these into Eq.(A.5) in Appendix A relating the coordinates in the orbit plane of the satellite to those in the reference plane in which the given coordinates and velocity are defined we have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = ABC \begin{pmatrix} r\cos f \\ r\sin f \\ 0 \end{pmatrix} \tag{B.27}$$

where

$$A = \begin{pmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{B.28}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{pmatrix} \tag{B.29}$$

$$C = \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{B.30}$$

and ω is the argument of pericentre. We can write Eq.(B.27) as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = AB \begin{pmatrix} r\cos(\omega+f) \\ r\sin(\omega+f) \\ 0 \end{pmatrix}$$

and so

$$\begin{pmatrix} r\cos(\omega+f) \\ r\sin(\omega+f) \\ 0 \end{pmatrix} = B^{-1}A^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{B.31}$$

where

$$B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{pmatrix} \tag{B.32}$$

$$A^{-1} = \begin{pmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{B.33}$$

In component form Eq.(B.31) is

$$r\cos(\omega+f) = x\cos\Omega + y\sin\Omega \tag{B.34}$$

$$r\sin(\omega+f) = -x\cos i\sin\Omega + y\cos i\cos\Omega + z\sin i \tag{B.35}$$

$$0 = x\sin i\sin\Omega - y\sin i\cos\Omega + z\cos i. \tag{B.36}$$

Eq.(B.36) just states **h**.**r**= 0 which is obvious from Eq.(B.16).

As the right-hand sides of Eqs.(B.34) and (B.35) are known then

$$\begin{aligned} w + f &= \tan^{-1}\left(\frac{\sin(\omega + f)}{\cos(\omega + f)}\right) \\ &= \tan^{-1}\left(\frac{-x\cos i \sin \Omega + y \cos i \cos \Omega + z \sin i}{x \cos \Omega + y \sin \Omega}\right). \end{aligned} \tag{B.37}$$

From Eqs.(B.25) and (B.26) we have

$$\begin{aligned} f &= \tan^{-1}\left(\frac{r \sin f}{r \cos f}\right) \\ &= \tan^{-1}\left(\frac{\sqrt{1-e^2}\sin E}{(\cos E - e)}\right). \end{aligned} \tag{B.38}$$

As f, $\omega + f$ and Ω are known from Eqs.(B.38), (B.37) and (B.24) respectively the longitude of pericentre can be found from

$$\varpi = \Omega + (\omega + f) - f. \tag{B.39}$$

From Eqs. (B.39) and (B.10) the mean longitude can now be calculated

$$\lambda = \varpi + l. \tag{B.40}$$

Hence the osculating elements a, e, i, λ, ϖ and Ω are calculated and they will be referred to the reference plane and axes in which the position and velocity are defined.

Appendix C

To compute the partial derivatives of position and velocity with respect to osculating elements from position and velocity, the mass of the primary and the mass of the satellite.

In deriving the partials we will need the osculating elements

$$\begin{aligned}
a &= \text{semi-major axis} \\
e &= \text{eccentricity} \\
i &= \text{inclination} \\
\lambda &= \text{mean longitude} \\
\varpi &= \text{longitude of the apse} \\
\Omega &= \text{longitude of the node.}
\end{aligned}$$

These are determined from formulae given in Appendix B. Also in Appendix B from Eq.(B.27) we have the coordinates related to the osculating elements

$$x = r(\cos\Omega\cos(\omega+f) - \sin\Omega\cos i\sin(\omega+f)) \quad (C.1)$$

$$y = r(\sin\Omega\cos(\omega+f) + \cos\Omega\cos i\sin(\omega+f)) \quad (C.2)$$

$$z = r\sin i\sin(\omega+f) \quad (C.3)$$

where r the radius vector and f the true anomaly are defined in Eqs.(B.2) and (B.38) respectively and $\omega = \varpi - \Omega$ the argument of pericentre.

We firstly consider partial derivatives of position with respect to osculating elements. The radius vector r in terms of the osculating elements is

$$r = a(1 - e\cos E) \quad (C.4)$$

where E, the eccentric anomaly, is determined from Eq.(B.15) in Appendix B. Then

$$\frac{\partial x}{\partial a} = \left(\frac{r}{a}\right)\left(\frac{x}{r}\right) = \frac{x}{a} \quad (C.5)$$

$$\frac{\partial y}{\partial a} = \left(\frac{r}{a}\right)\left(\frac{y}{r}\right) = \frac{y}{a} \quad (C.6)$$

$$\frac{\partial z}{\partial a} = \left(\frac{r}{a}\right)\left(\frac{z}{r}\right) = \frac{z}{a}. \quad (C.7)$$

Next consider partials with respect to the eccentricity. From Eqs.(C.1),(C.2) and (C.3) we have

$$\frac{\partial x}{\partial e} = \frac{\partial r}{\partial e}\left(\frac{x}{r}\right) + r\frac{\partial f}{\partial e}(-\cos\Omega\sin(\omega+f) - \sin\Omega\cos i\cos(\omega+f)) \quad (C.8)$$

$$\frac{\partial y}{\partial e} = \frac{\partial r}{\partial e}\left(\frac{y}{r}\right) + r\frac{\partial f}{\partial e}(-\sin\Omega\sin(\omega+f) + \cos\Omega\cos i\cos(\omega+f)) \quad (C.9)$$

$$\frac{\partial z}{\partial e} = \frac{\partial r}{\partial e}\left(\frac{z}{r}\right) + r\frac{\partial f}{\partial e}\sin i\cos(\omega+f). \quad (C.10)$$

From Eq.(C.4) and Kepler's equation

$$E - e \sin E = l \tag{C.11}$$

where $l = \lambda - \varpi$ is the mean anomaly we have

$$\frac{\partial r}{\partial e} = \frac{a^2(e - \cos E)}{r}. \tag{C.12}$$

From the equation relating the true anomaly to the eccentric anomaly

$$\tan \frac{f}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \tag{C.13}$$

we obtain

$$\frac{\partial f}{\partial e} = \frac{a\sqrt{1-e^2} \sin E}{r} \left(\frac{1}{1-e^2} + \frac{a}{r} \right). \tag{C.14}$$

Hence partials with respect to eccentricity are computed from Eqs.(C.8), (C.9), (C.10), (C.12) and (C.14). The partials with respect to the mean anomaly are computed analogously. We have

$$\frac{\partial x}{\partial l} = \frac{\partial r}{\partial l} \left(\frac{x}{r} \right) + r \frac{\partial f}{\partial l} (-\cos \Omega \sin(\omega + f) - \sin \Omega \cos i \cos(\omega + f)) \tag{C.15}$$

$$\frac{\partial y}{\partial l} = \frac{\partial r}{\partial l} \left(\frac{y}{r} \right) + r \frac{\partial f}{\partial l} (-\sin \Omega \sin(\omega + f) + \cos \Omega \cos i \cos(\omega + f)) \tag{C.16}$$

$$\frac{\partial z}{\partial l} = \frac{\partial r}{\partial l} \left(\frac{z}{r} \right) + r \frac{\partial f}{\partial l} \sin i \cos(\omega + f) \tag{C.17}$$

where from Eqs.(C.4), (C.11) and (C.13)

$$\frac{\partial r}{\partial l} = \frac{a^2 e \sin E}{r} \tag{C.18}$$

$$\frac{\partial f}{\partial l} = \frac{a^2 \sqrt{(1-e^2)}}{r^2}. \tag{C.19}$$

The partials with respect to ω, i and Ω follow from straightforward differentiation of Eqs.(C.1), (C.2) and (C.3). We have

$$\frac{\partial x}{\partial \omega} = r(-\cos \Omega \sin(\omega + f) - \sin \Omega \cos i \cos(\omega + f)) \tag{C.20}$$

$$\frac{\partial y}{\partial \omega} = r(-\sin \Omega \sin(\omega + f) + \cos \Omega \cos i \cos(\omega + f)) \tag{C.21}$$

$$\frac{\partial z}{\partial \omega} = r \sin i \cos(\omega + f) \tag{C.22}$$

$$\frac{\partial x}{\partial i} = z \sin \Omega \qquad (C.23)$$
$$\frac{\partial y}{\partial i} = -z \cos \Omega \qquad (C.24)$$
$$\frac{\partial z}{\partial i} = z \cot i \qquad (C.25)$$

$$\frac{\partial x}{\partial \Omega} = -y \qquad (C.26)$$
$$\frac{\partial y}{\partial \Omega} = x \qquad (C.27)$$
$$\frac{\partial z}{\partial \Omega} = 0. \qquad (C.28)$$

Next we consider partial derivatives of velocity with respect to osculating elements. Differentiating with respect to time Eqs.(C.1), (C.2) and (C.3) we obtain

$$\dot{x} = \dot{r}(\cos \Omega \cos(\omega + f) - \sin \Omega \cos i \sin(\omega + f)) - r\dot{f}(\cos \Omega \sin(\omega + f) + \sin \Omega \cos i \cos(\omega + f)) \quad (C.29)$$
$$\dot{y} = \dot{r}(\sin \Omega \cos(\omega + f) + \cos \Omega \cos i \sin(\omega + f)) - r\dot{f}(\sin \Omega \sin(\omega + f) - \cos \Omega \cos i \cos(\omega + f)) \quad (C.30)$$
$$\dot{z} = \dot{r} \sin i \sin(\omega + f) + r\dot{f} \sin i \cos(\omega + f). \qquad (C.31)$$

We calculate $\dot{r}$ from Eq.(C.4) using from Appendix B Eq.(B.11) and obtain

$$\dot{r} = \frac{na^2 e \sin E}{r} \qquad (C.32)$$

where n is the mean motion.

To calculate $\dot{f}$ in Eqs.(C.29), (C.30) and (C.31) we use the equation of an ellipse in the form

$$r = \frac{a(1-e^2)}{1 + e \cos f} \qquad (C.33)$$

which on differentiating we find

$$\dot{f} = \frac{\dot{r} a(1-e^2)}{r^2 e \sin f}$$

and on substituting for $\dot{r}$ from Eq.(C.32)

$$\dot{f} = \frac{na^3(1-e^2)}{r^3}\left(\frac{\sin E}{\sin f}\right). \qquad (C.34)$$

From Appendix B Eq.(B.26) we have

$$\frac{\sin E}{\sin f} = \frac{r}{a\sqrt{(1-e^2)}} \qquad (C.35)$$

and so from Eqs.(C.34) and (C.35)

$$\dot{f} = \frac{na^2\sqrt{(1-e^2)}}{r^2}. \qquad (C.36)$$

From Appendix B Eq.(B.12) we obtain

$$\frac{\partial n}{\partial a} = -\frac{3n}{2a}. \qquad (C.37)$$

From Eqs. (C.32) and (C.37) we obtain

$$\frac{\partial \dot{r}}{\partial a} = -\frac{\dot{r}}{2a} \qquad (C.38)$$

and from Eqs. (C.4), (C.36) and (C.37)

$$\frac{\partial (r\dot{f})}{\partial a} = -\frac{r\dot{f}}{2a}. \qquad (C.39)$$

From Eqs. (C.29), (C.30), (C.31), (C.38) and (C.39) we have

$$\frac{\partial \dot{x}}{\partial a} = -\frac{\dot{x}}{2a} \qquad (C.40)$$

$$\frac{\partial \dot{y}}{\partial a} = -\frac{\dot{y}}{2a} \qquad (C.41)$$

$$\frac{\partial \dot{z}}{\partial a} = -\frac{\dot{z}}{2a}. \qquad (C.42)$$

To calculate the partial derivatives with respect to e and l we note in Eqs.(C.29), (C.30) and (C.31) only r, f, $\dot{r}$ and $\dot{f}$ are functions of e and l. Let $\sigma = e$ or l then

$$\begin{aligned}\frac{\partial \dot{x}}{\partial \sigma} =& \left(\frac{\partial \dot{r}}{\partial \sigma} - r\dot{f}\frac{\partial f}{\partial \sigma}\right)(\cos\Omega\cos(\omega+f) - \sin\Omega\cos i\sin(\omega+f)) \\ &+ \left(\dot{r}\frac{\partial f}{\partial \sigma} + \frac{\partial (r\dot{f})}{\partial \sigma}\right)(-\cos\Omega\sin(\omega+f) - \sin\Omega\cos i\cos(\omega+f)) \qquad (C.43)\end{aligned}$$

$$\begin{aligned}\frac{\partial \dot{y}}{\partial \sigma} =& \left(\frac{\partial \dot{r}}{\partial \sigma} - r\dot{f}\frac{\partial f}{\partial \sigma}\right)(\sin\Omega\cos(\omega+f) + \cos\Omega\cos i\sin(\omega+f)) \\ &+ \left(\dot{r}\frac{\partial f}{\partial \sigma} + \frac{\partial (r\dot{f})}{\partial \sigma}\right)(-\sin\Omega\sin(\omega+f) + \cos\Omega\cos i\cos(\omega+f)) \qquad (C.44)\end{aligned}$$

$$\frac{\partial \dot{z}}{\partial \sigma} = \left(\frac{\partial \dot{r}}{\partial \sigma} - r\dot{f}\frac{\partial f}{\partial \sigma}\right)\sin i\sin(\omega+f) + \left(\dot{r}\frac{\partial f}{\partial \sigma} + \frac{\partial (r\dot{f})}{\partial \sigma}\right)\sin i\cos(\omega+f). \qquad (C.45)$$

From Eq.(C.32) using Eqs.(C.4) and (C.11) we have

$$\frac{\partial \dot{r}}{\partial e} = \frac{na^4\sin E(1-e^2)}{r^3}. \qquad (C.46)$$

From Eqs.(C.14) and (C.36)

$$r\dot{f}\frac{\partial f}{\partial e} = \frac{na^3(1-e^2)}{r^2}\sin E\left(\frac{1}{(1-e^2)} + \frac{a}{r}\right) \qquad (C.47)$$

and so from Eqs. (C.46) and (C.47)

$$\frac{\partial \dot{r}}{\partial e} - r\dot{f}\frac{\partial f}{\partial e} = -\frac{na^3 \sin E}{r^2}. \qquad (C.48)$$

Next differentiating Eq. (C.36) with respect to e and using Eqs. (C.4) and (C.11) we have

$$\frac{\partial \dot{f}}{\partial e} = \frac{2na^4\sqrt{(1-e^2)}(\cos E - e)}{r^4} - \frac{na^2 e}{\sqrt{(1-e^2)}r^2}. \qquad (C.49)$$

From Eqs. (C.12), (C.14), (C.32), (C.36) and (C.49) we have

$$\dot{r}\frac{\partial f}{\partial e} + \frac{\partial (r\dot{f})}{\partial e} = \frac{na^2 \cos E}{r\sqrt{(1-e^2)}}. \qquad (C.50)$$

Thus partial derivatives with respect to e can be calculated from Eqs. (C.43), (C.44) and (C.45) where $\sigma = e$ using Eqs. (C.48) and (C.50).

Differentiating Eq. (C.32) with respect to l using Eqs. (C.4) and (C.11) we have

$$\frac{\partial \dot{r}}{\partial l} = -\frac{na^4 e^2 \sin^2 E}{r^3} + \frac{na^3 e \cos E}{r^2}. \qquad (C.51)$$

From Eqs. (C.19), (C.36) and (C.51)

$$\frac{\partial \dot{r}}{\partial l} - r\dot{f}\frac{\partial f}{\partial l} = -\frac{na^3}{r^2}. \qquad (C.52)$$

Next differentiating Eq. (C.36) with respect to l and using Eqs. (C.4) and (C.11) we have

$$\frac{\partial \dot{f}}{\partial l} = -\frac{2na^4 e \sin E\sqrt{(1-e^2)}}{r^4}\ . \qquad (C.53)$$

From Eqs. (C.18), (C.19), (C.32), (C.36) and (C.53) we find

$$\dot{r}\frac{\partial f}{\partial l} + \frac{\partial (r\dot{f})}{\partial l} = 0. \qquad (C.54)$$

Thus partial derivatives with respect to l can be calculated from Eqs. (C.43), (C.44) and (C.45) where $\sigma = l$ using Eqs. (C.52) and (C.54).

The partial derivatives with respect to ω, i and Ω follow from straightforward differentiation of Eqs. (C.29), (C.30) and (C.31). We have

$$\frac{\partial \dot{x}}{\partial \omega} = \dot{r}(-\cos \varOmega \sin(\omega + f) - \sin \varOmega \cos i \cos(\omega + f)) - r\dot{f}(\cos \varOmega \cos(\omega + f) - \sin \varOmega \cos i \sin(\omega + f)) \quad (C.55)$$

$$\frac{\partial \dot{y}}{\partial \omega} = \dot{r}(-\sin \varOmega \sin(\omega + f) + \cos \varOmega \cos i \cos(\omega + f)) - r\dot{f}(\sin \varOmega \cos(\omega + f) + \cos \varOmega \cos i \sin(\omega + f)) \quad (C.56)$$

$$\frac{\partial \dot{z}}{\partial \omega} = \dot{r} \sin i \cos(\omega + f) - r\dot{f} \sin i \sin(\omega + f) \quad (C.57)$$

$$\frac{\partial \dot{x}}{\partial i} = \dot{z} \sin \varOmega \quad (C.58)$$

$$\frac{\partial \dot{y}}{\partial i} = -\dot{z} \cos \varOmega \quad (C.59)$$

$$\frac{\partial \dot{z}}{\partial i} = \dot{z} \cot i \quad (C.60)$$

$$\frac{\partial \dot{x}}{\partial \varOmega} = -\dot{y} \quad (C.61)$$

$$\frac{\partial \dot{y}}{\partial \varOmega} = \dot{x} \quad (C.62)$$

$$\frac{\partial \dot{z}}{\partial \varOmega} = 0. \quad (C.63)$$

Finally, transforming the partial derivatives with respect to l and ω to partial derivatives with respect to the mean longitude λ and the longitude of pericentre ϖ respectively we have

$$\frac{\partial(x, y, z, \dot{x}, \dot{y}, \dot{z})}{\partial \lambda} = \frac{\partial(x, y, z, \dot{x}, \dot{y}, \dot{z})}{\partial l} \quad (C.64)$$

$$\frac{\partial(x, y, z, \dot{x}, \dot{y}, \dot{z})}{\partial \varpi} = \frac{\partial(x, y, z, \dot{x}, \dot{y}, \dot{z})}{\partial \omega} - \frac{\partial(x, y, z, \dot{x}, \dot{y}, \dot{z})}{\partial l} \quad (C.65)$$

with partial derivatives with respect to the longitude of the node N now computed from

$$\frac{\partial(x, y, z, \dot{x}, \dot{y}, \dot{z})}{\partial N} = \frac{\partial(x, y, z, \dot{x}, \dot{y}, \dot{z})}{\partial \varOmega} - \frac{\partial(x, y, z, \dot{x}, \dot{y}, \dot{z})}{\partial \omega}. \quad (C.66)$$