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Compact Ephemerides for Differential Tangent Plane Coordinates of Planetary Satellites

by

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Summary

Formulae and methods are given for two different ways of representing differential tangent plane coordinates of planetary satellites, one uses Chebyshev polynomials, the other uses a quasi Fourier representation involving mixed functions of secular and periodic terms. The precision of the approximating function in each case being $0\overset{''}{.}01$. The resulting ephemerides will enable observations to be compared with precise ephemerides as well as for telescope setting. Calculations on the satellites of Saturn prove that the total number of coefficients necessary to describe one year of the motion is always smaller in the case of the mixed functions. It is proposed that the current method of representing the positions of planetary satellites in *The Astronomical Almanac* is replaced by the mixed function method.

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1. Introduction

The satellite section of *The Astronomical Almanac* has evolved over a period of time to its present form for historical reasons. A variety of geocentric phenomena and dynamical information are given. These include data on superior geocentric conjunctions and geocentric phenomena of the Galilean satellites and differential coordinates for all the other major satellites, often computed using tables of times of elongation and apparent distance and position angle. In addition, for the eight major satellites of Saturn, orbital elements are given at 4-day intervals. The orbital elements were first included in *The Nautical Almanac* for 1935 and were intended to facilitate the prediction of eclipses, transits, shadow transits and occultations for the passage of the Earth through the ring-plane in 1936. The satellite data in the Almanac at present is primarily used for planning observations and identification on photographic plates or CCD frames. The satellite section is now in the process of being reorganised and the data given will be derived from the best available current theories. Thus, in addition to planning observations and identification, they will also provide a means of determining the quality of observations through the computation of residuals.

The major problem of publishing satellite ephemerides is to present the data as compactly as possible and still keep the precision of the original theory. The problem is further complicated by the large range of orbital periods e.g. in the Saturnian system the period of Iapetus is about 79 days whereas for Mimas it is less than one day. To accommodate, in particular the fast-moving satellites, and to have compactness whilst maintaining precision, different methods of representation must be sought from the usual standard tables giving positions for equal time intervals. Two methods are briefly described to represent compactly the differential tangent plane coordinates. The first is based on the well-known Chebyshev polynomials whilst the other uses a novel approach formulated by Chapront and Vu (1984) from a suggestion by Bacchus(1981) in which the coordinates are represented by a mixture of periodic and secular functions. The latter method is particularly suitable for representing coordinates of Solar System bodies because of their quasi-periodic nature. This form of representation of ephemerides has become well established in the annual publication of the Bureau des Longitudes "Ephémérides des Satellites de Mars, Jupiter, Saturne et Uranus".

Ephemerides have been determined for the eight major satellites of Saturn using the Chebyshev approximation and mixed functions representation. The different methods are compared and their various merits discussed.

The differential tangent plane coordinates (X and Y) of the satellites with respect to the centre of mass of the planet are given by the exact expressions

$$X = \frac{\cos\delta_{\rm S}\sin(\alpha_{\rm S}-\alpha_{\rm P})}{\sin\delta_{\rm S}\sin\delta_{\rm P}+\cos\delta_{\rm S}\cos\delta_{\rm P}\cos(\alpha_{\rm S}-\alpha_{\rm P})} \tag{1}$$

$$Y = \frac{\sin\delta_{\rm S}\cos\delta_{\rm P}-\cos\delta_{\rm S}\sin\delta_{\rm P}\cos(\alpha_{\rm S}-\alpha_{\rm P})}{\sin\delta_{\rm S}\sin\delta_{\rm P}+\cos\delta_{\rm S}\cos\delta_{\rm P}\cos(\alpha_{\rm S}-\alpha_{\rm P})}$$

where $\alpha_{\rm P}$, $\delta_{\rm P}$ are the right ascension (RA) and declination (Dec) of the planet and $\alpha_{\rm S}$, $\delta_{\rm S}$ the RA and Dec of the satellite. The Y-axis is set towards the pole of the equator (North) and the X-axis in the direction of increasing right ascension (East).

Astronomers require the observables $\Delta\alpha\cos\delta_{\rm P}$ and $\Delta\delta$ where $\Delta\alpha = \alpha_{\rm S}-\alpha_{\rm P}$ and $\Delta\delta = \delta_{\rm S}-\delta_{\rm P}$. Since $\Delta\alpha$ and $\Delta\delta$ are small, to the first order in small quantities

$$\Delta\alpha\cos\delta_{\rm P} = X \tag{2}$$

$$\Delta\delta = Y$$

and to the second order

$$\Delta\alpha \cos\delta_{\mathrm{p}} = X + \sin 1'' XY \tan\delta_{\mathrm{p}} \tag{3}$$
$$\Delta\delta = Y - \sin 1'' \frac{X^2}{2} \tan\delta_{\mathrm{p}}$$

Formulae (3) are needed for satellites with large elongation, such as Iapetus.

We will now describe how compact ephemerides can be constructed for the coordinates X and Y first using a Chebyshev representation and second a quasi Fourier representation.

2. **Chebyshev representation**

Let $f(t)$ be a differential tangent plane coordinate defined over the interval $[t_0, t_0 + \Delta t]$, where the interval Δt will be determined for each satellite. Chebyshev polynomials are defined over the interval $[-1, 1]$, and we change to this interval using the transformation:

$$x = \frac{2(t - t_0)}{\Delta t} - 1. \tag{4}$$

This transforms the function $f(t)$, $t \in [t_0, t_0 + \Delta t]$ to $f(x)$, $x \in [-1, 1]$, see Fig. 1.

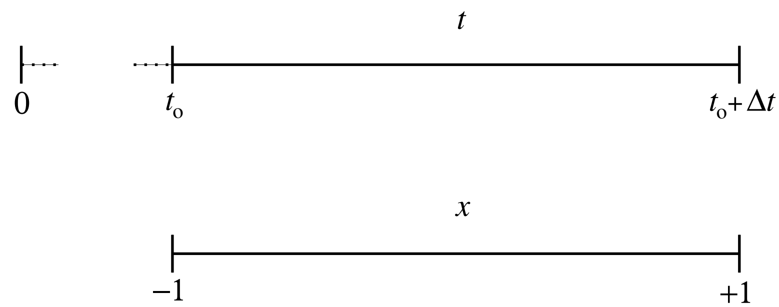


Fig. 1 Comparison of the t-interval with the x-interval using transformation given by equation (4).

We are going to represent the function $f(x)$ by the approximation

$$p_n(x) = {\sum_{r=0}^{n}}' c_r T_r(x) \tag{5}$$

for some n, where c_r are constants to be determined and $T_r(x)$ are Chebyshev polynomials of degree r, defined by

$$T_r(x) = \cos(r \cos^{-1} x), \quad -1 \leq x \leq 1.$$

The prime on the summation sign indicating the first term in the summation is to be halved. A property of Chebyshev polynomials is that

$$\sum_{j=0}^{N}{}'' T_r(x_j) T_s(x_j) = \begin{cases} N & (\mathrm{r = s = 0 \ or \ N}) \\ N/2 & (\mathrm{r = s \neq 0 \ or \ N}) \\ 0 & (\mathrm{r \neq s}) \end{cases} \tag{6.1}$$

where

$$x_j = \cos\frac{\pi j}{N}. \tag{6.2}$$

The double prime on the summation sign indicating the first and last term in the summation are to be halved.

The $c_r, r = 0, 1, \ldots, n$ are determined by minimising

$$S_c = \sum_{j=0}^{N}{}'' (f(x_j) - p_n(x_j))^2. \tag{7}$$

For (7) to be a minimum

$$\frac{\partial S_c}{\partial c_0} = 0, \frac{\partial S_c}{\partial c_1} = 0, \ldots, \frac{\partial S_c}{\partial c_n} = 0. \tag{8}$$

These will be a set of $n+1$ equations in the $n+1$ unknowns c_r and are the normal equations. Taking $n < N$ and using (6.1) the equations (8) yield

$$c_r = \frac{2}{N} \sum_{j=0}^{N}{}'' f(x_j) T_r(x_j), \quad r = 0, 1, \ldots, n. \tag{9}$$

Hence in the Chebyshev approximation, for $n < N$

$$f(x) \approx \sum_{r=0}^{n}{}' c_r T_r(x) \tag{10}$$

where c_r are given by (9) and the x_j by (6.2).

In practice we evaluate (9) for points $x_j = \cos \pi j/N$ and then truncate at coefficient c_n the series representing $f(x)$. If a precision of ε is required then as $T_r(x)$ reaches its greatest absolute value of unity at points $x = x_j$ in interval $[-1, 1]$ we must have

$$|c_{n+1}| + |c_{n+2}| + \ldots + |c_N| \leq \varepsilon. \tag{11}$$

The evaluation of the Chebyshev series (10) is made using the recurrence relation

$$\sum_{r=0}^{n}{}' c_r T_r(x) = \frac{c_0}{2} + x b_1(x) - b_2(x) \tag{12.1}$$

where

$$\begin{aligned} b_r(x) &= 2x b_{r+1}(x) - b_{r+2}(x) + c_r \\ b_{n+1}(x) &= b_{n+2}(x) = 0. \end{aligned} \tag{12.2}$$

3. Mixed functions representation

Suppose again $f(t)$ be a differential tangent plane coordinate defined on the interval $[t_0, t_0 + \Delta t]$. This can be mapped to the interval $[-1, 1]$ using the linear transformation (4). The problem is now one of representing compactly the behaviour of $f(x)$ on $[-1, 1]$ taking into account the quasi-periodic character of $f(x)$.

Following Arlot et al (1986) we take as a set of basis functions on $[-1, 1]$

$$\begin{aligned} \{\phi_1, \phi_2, \ldots, \phi_{10}\} = \{&1, \cos \omega x, \cos 2\omega x, x \sin \omega x, x^2 \cos \omega x; \\ &x, \sin \omega x, \sin 2\omega x, x \cos \omega x, x^2 \sin \omega x\} \end{aligned} \tag{13}$$

where

$$\omega = \frac{\nu \Delta t}{2} \tag{14}$$

and ν is the main orbital frequency of the motion of the satellite. The first five functions in (13) are even ones ($g(x) = g(-x)$) and the last five functions odd ($g(x) = -g(-x)$).

We now represent $f(x)$ approximately by $F(x)$ where

$$F(x) = \sum_{i=1}^{10} q_i \phi_i(x) \tag{15}$$

where the q_i are numbers to be determined.

The q_i are determined by making

$$S_m = \left[\int_{-1}^{+1} (f(x) - F(x))^2 \, dx \right]^{\frac{1}{2}} \tag{16}$$

a minimum.

From (15) and (16) we have

$$S_m = \left[\int_{-1}^{+1} \left(f(x) - (q_1\phi_1 + q_2\phi_2 + \ldots + q_{10}\phi_{10}) \right)^2 dx \right]^{\frac{1}{2}} \tag{17}$$

and for this to be a minimum

$$\frac{\partial S_m}{\partial q_1} = 0, \frac{\partial S_m}{\partial q_2} = 0, \ldots, \frac{\partial S_m}{\partial q_{10}} = 0. \tag{18}$$

In practice we are making a continuous least-squares fit of $F(x)$ to $f(x)$ and the equations (18) are the normal equations.

Now from theory $f(x)$ and the $\phi_i, i = 1, 2 \ldots, 10$ are functions which are square summable so

$$\int_{-1}^{+1} \left(f(x) - (q_1\phi_1 + \ldots + q_{10}\phi_{10}) \right)^2 dx$$

has a finite positive value. The normal equations are therefore

$$\int_{-1}^{+1} (f(x) - (q_1\phi_1 + \ldots + q_{10}\phi_{10}))\phi_i \, dx = 0, \;\; i = 1, 2, \ldots, 10. \tag{19}$$

Let

$$m_{i,j} = \int_{-1}^{+1} \phi_i(x)\phi_j(x) \, dx \tag{20}$$

then (19) can be written in matrix form

$$Mq = b \tag{21}$$

where M is a symmetrical 10×10 matrix with elements $m_{i,j}$ and

$$q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_{10} \end{pmatrix} \tag{22}$$

$$b = \begin{pmatrix} \int_{-1}^{+1} f(x)\phi_1(x) \, dx \\ \int_{-1}^{+1} f(x)\phi_2(x) \, dx \\ \vdots \\ \int_{-1}^{+1} f(x)\phi_{10}(x) \, dx \end{pmatrix}. \tag{23}$$

The matrix M can be simplified. If $g(x)$ is an even function

$$\int_{-1}^{+1} g(x)\,dx = 2\int_{0}^{1} g(x)\,dx$$

and if $g(x)$ is an odd function

$$\int_{-1}^{+1} g(x)\,dx = 0.$$

Now we recall from (13) $\phi_1, \phi_2 \ldots, \phi_5$ are even functions and $\phi_6, \phi_7, \ldots, \phi_{10}$ are odd functions. Clearly the product of two even functions or two odd functions is an even function and the product of an odd function and an even function is an odd function. Thus

$$\int_{-1}^{+1} \phi_i(x)\phi_j(x)\,dx = 0, \text{ for } i = 1, 2, \ldots, 5, \; j = 6, 7, \ldots, 10 \tag{24}$$

and the matrix M is of the form

$$\begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & m_{1,5} & 0 & 0 & 0 & 0 & 0 \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} & m_{2,5} & 0 & 0 & 0 & 0 & 0 \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} & m_{3,5} & 0 & 0 & 0 & 0 & 0 \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} & m_{4,5} & 0 & 0 & 0 & 0 & 0 \\ m_{5,1} & m_{5,2} & m_{5,3} & m_{5,4} & m_{5,5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_{6,6} & m_{6,7} & m_{6,8} & m_{6,9} & m_{6,10} \\ 0 & 0 & 0 & 0 & 0 & m_{7,6} & m_{7,7} & m_{7,8} & m_{7,9} & m_{7,10} \\ 0 & 0 & 0 & 0 & 0 & m_{8,6} & m_{8,7} & m_{8,8} & m_{8,9} & m_{8,10} \\ 0 & 0 & 0 & 0 & 0 & m_{9,6} & m_{9,7} & m_{9,8} & m_{9,9} & m_{9,10} \\ 0 & 0 & 0 & 0 & 0 & m_{10,6} & m_{10,7} & m_{10,8} & m_{10,9} & m_{10,10} \end{pmatrix}. \tag{25}$$

As matrix M is symmetric, only the upper triangular elements need to be evaluated and these are given explicitly in Appendix A.

The integrals (23) must be evaluated numerically. A simple and rapid method to compute these is to use Gaussian quadrature. Formulae describing this method are given in Appendix B.

From (21)

$$q = M^{-1}b. \tag{26}$$

Thus the approximation $F(x)$ to $f(x)$ can be found from (15). Finally, $F(t)$ can be found using equations (4) and (14).

A maximum error at the ends of the interval is a common feature of the fitting process described here. A simple way to overcome this difficulty is to choose a time interval, when evaluating the coefficients q_i, slightly longer than the interval required. Thus to write $F(x)$ in terms of the original independent variable t we use the linear transformation

$$x = \frac{2(t + t_c)}{\Delta t} - 1 \tag{27}$$

where t_c is the value for the margin at the ends and Δt the length of the complete interval. The origin is now $t_0 + t_c$. The interval of use is now $[t_0 + t_c, t_0 + \Delta t - t_c]$, see Fig. 2.

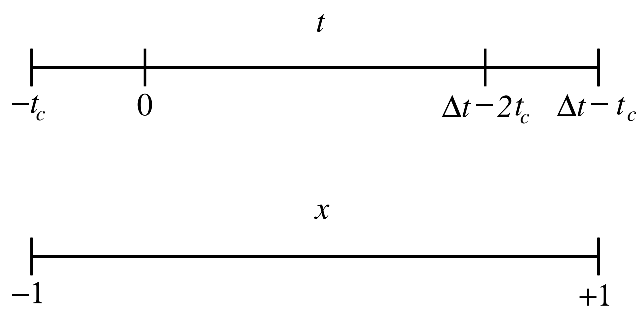


Fig. 2 Comparison of the t-interval with the x-interval using transformation given by equation (27).

Thus substituting x from (27) and ω from (14) into (15) and simplifying we obtain

$$f(t) \approx F(t) = a_0 + a_1 t + b_1 \sin(\nu t + \phi_1) + b_2 \sin(2\nu t + \phi_2) + b_3 t \sin(\nu t + \phi_3) + b_4 t^2 \sin(\nu t + \phi_4) \qquad (28)$$

where the amplitudes and phases are functions of $q_1, q_2, \ldots, q_{10}, t_c$ and Δt. In Appendix C expressions are given for $a_0, a_1, b_1, \phi_1, b_2, \phi_2, b_3, \phi_3, b_4$ and ϕ_4 in terms of $q_1, q_2, \ldots, q_{10}, t_c$and Δt.

4. Programs

The program for the method based on mixed functions is called **diffmixfn.for**, and the program based on Chebyshev polynomials is called **diffcheby.for**. They enable the differential tangent plane coordinates to be represented over specified intervals. A program called **diffephem.for** is also available which obtains from the source ephemeris coordinates for a given date to compare with these representations. For each of these source programs there is an input file **diffmixfn.dat**, **diffcheby.dat** and **diffephem.dat** and a corresponding output file **diffmixfn.out**, **diffcheby.out** and **diffephem.out**, respectively. These are discussed more fully later.

The planetary positions are obtained from the latest JPL ephemeris ($DE200$ or $DE403$) which resides in a file on-line. In the computations described here the $DE200$ ephemeris was used. The satellite theories are in program **theories.for** which are linked to the source programs at compilation. Constants of the theories are stored in files **saturnsat.elm** for the Saturnian satellites, **marssat.elm** for the Martian satellites etc. A light-time correction is applied to the planet and satellite ephemerides. The differential tangent plane coordinates are output in the mean equatorial frame of $J2000$, the reference frame of $DE200$. Many of the satellite theories are given in the $B1950$ system and so the coordinates have to be transformed to the $J2000$ system.

In the case of the mixed functions representation the order of the Gaussian n-point formula will in general be larger the greater the angular frequency of the satellite, although it is also dependent on the length of interval for fitting. File **legpolzs.lgz**, used in **diffmixfn.for**, contains the zeros of the Legendre polynomials and corresponding weights. Files **legpolzs.lgz30**, **legpolzs.lgz60**, **legpolzs.lgz125** and **legpolzs.lgz250** contain zeros and weights of Legendre polynomials up to and including orders 30, 60, 125 and 250 respectively. Depending on which satellite coordinates are to be computed the appropriate file must be copied to **legpolzs.lgz**. File **legpolzs.dat**, also used by **diffmixfn.for**, must contain the order of the largest Legendre polynomial for which data are contained in **legpolzs.lgz**, i.e. it must be 30,60,125 or 250. If zeros and weights of higher order Legendre polynomials are needed program **legpolzs.for** will provide them by putting the desired order into input file **legpolzs.dat**. The quadratures are evaluated for successive Gaussian n-point formulae until the value of the quadratures from the n and $(n+1)$th point formula differ by a preassigned accuracy.

In **diffmixfn.for** the matrix inversion is carried out using the standard Gauss-Jordan method.

The parameters in the input files for **diffmixfn.for**, **diffcheby.for** and **diffephem.for** have the following identification:-

diffmixfn.dat

ISAT - planet number

JSAT - satellite number of planet ISAT

IYR - year for which coefficients required

TSPAN - length of complete interval (in days) used for fitting

TC - end interval (in days)

NU - associated frequency of satellite (rads/d)

EPS - accuracy for quadratures

NUMBER - number of coefficients not solved for

IKP(10) - indicator for which coefficients not to be solved for

diffcheby.dat

ISAT - planet number

JSAT - satellite number of planet ISAT

IYR - year for which coefficients required

NDAY - length of interval (in days) used for fitting

NC - number of Chebyshev coefficients for each interval

EPS - accuracy criterion for truncation of Chebyshev series

diffephem.dat

ISAT - planet number

JSAT - satellite number of planet ISAT

IYR - year for which coordinates are required

TINC - interval for which coordinates are to be computed

The output files contain:-

diffmixfn.out

Coefficients of mixed functions representation for intervals of length TSPAN$-2\times$TC for satellite JSAT of planet ISAT for year IYR.

diffcheby.out

Chebyshev coefficients for intervals of length NDAY for satellite JSAT of planet ISAT for year IYR.

diffephem.out

Coordinates of satellite JSAT of planet ISAT at intervals of TINC for year IYR.

The coordinates computed from the mixed functions representation and Chebyshev representation are compared with those from the source ephemeris in program **checkdiff.for**. The comparison will be at intervals of TINC days. The output file **checkdiff.out** contains the differences in ascending order and so the largest differences can be easily found.

5. **Results**

The programs described in the previous section were used to compute ephemerides of differential tangent plane coordinates for one year (1993 was used) for the eight major satellites of Saturn. The theory used for Hyperion was from Taylor (1992). The theories used for the other satellites were those described in Taylor and Shen (1988) but the mean motions and secular rates were taken from Dourneau (1987). The orbital elements were determined by fitting the theories to observations made in the period 1967 to 1983.

For the Chebyshev representation the length of intervals for fitting and the number of coefficients used were taken from Arlot *et al* (1986). For the method based on mixed functions the length of interval and the number of coefficients were taken from the Bureau des Longitudes publication "Ephémérides des Satellites de Mars, Jupiter, Saturne et Uranus pour 1993". The fitting parameters were chosen so that the precision of the fit in each case is about 0.″01. Table 1 gives the orbital period, natural frequency and frequency used in the mixed functions representation. The frequency ν used is the same as the natural frequency of the satellite (to 3 decimal places) with the exception of Hyperion. For Hyperion the frequency ν is chosen close to Titan's natural frequency. The fitting parameters are given in Table 2. The quadratures were evaluated for successive Gaussian n-point formulae until the value of the quadratures from the n and $(n+1)^{th}$ point formula differed by less than a pre-assigned value. In our evaluations of the quadratures an accuracy of 1×10^{-8} was chosen. From Table 2 the choice of the Gaussian n-point formula is clearly seen to be dependent on the value of ω. This should act as a guide when determining coefficients in the mixed function representation for satellites of other planetary systems.

Table 1 Natural and associated frequencies

Satellite	**Orbital period** days	**Natural frequency** radians/d	**Associated frequency, ν** radians/d
Mimas	0.9	6.66706170	6.667
Enceladus	1.4	4.58553671	4.586
Tethys	1.9	3.32830645	3.328
Dione	2.7	2.29571764	2.296
Rhea	4.5	1.39085371	1.391
Titan	15.9	0.39404261	0.394
Hyperion	21.3	0.29530886	0.394
Iapetus	79.3	0.07920219	0.079

Table 2 Fitting parameters

Satellite	**Orbital Period**	**Chebyshev**		**Mixed functions representation**				
		Length of interval	No. of coeffts. for X or Y	Length of interval ΔT	Overlap interval used	No. of coeffts. for X or Y	Freq. $\omega = \frac{\nu \Delta T}{2}$	Largest n-point Gaussian formula used
	days	days		days	days		radians	
Mimas	0.9	0.5	9	4	0.1	10	13.334	60
Enceladus	1.4	1	9	16	0.4	10	36.688	125
Tethys	1.9	1	8	16	0.4	10	26.624	77
Dione	2.7	2	9	16	0.4	10	18.368	64
Rhea	4.5	3	9	16	0.4	10	11.128	39
Titan	15.9	8	9	11	0.2	8	2.167	15
Hyperion	21.3	8	9	8	0.2	8	1.576	13
Iapetus	79.3	16	9	16	0.4	6	0.632	12

The coefficients in the mixed function representation for some satellites were highly correlated. This was particularly so for Titan, Hyperion and Iapetus, where the normal matrix, M, has a determinant near to zero. Although small changes in the differential tangent plane coordinates produced large changes in the coefficients for the mixed functions, the expressions for these mixed functions are equally valid.

Fig 3a Mimas: Run−off of approximation from source for X − interval of use 4d to 8d

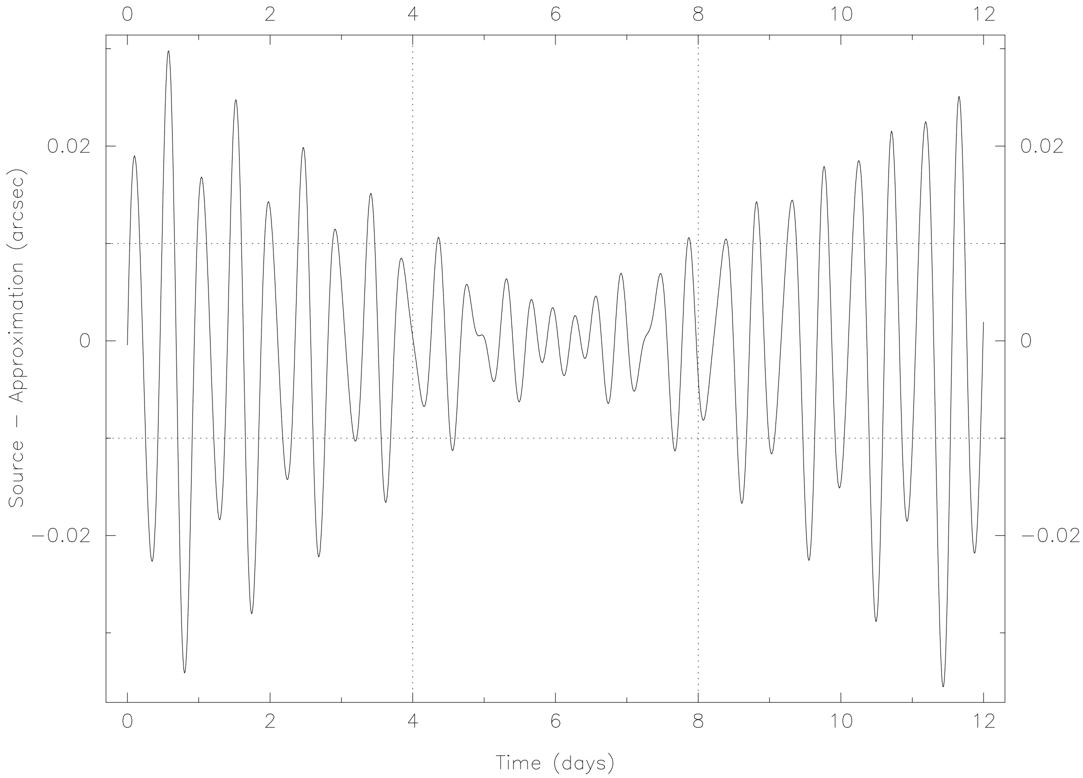


Fig 3b Enceladus: Run−off of approximation from source for X − interval of use 16d to 32d

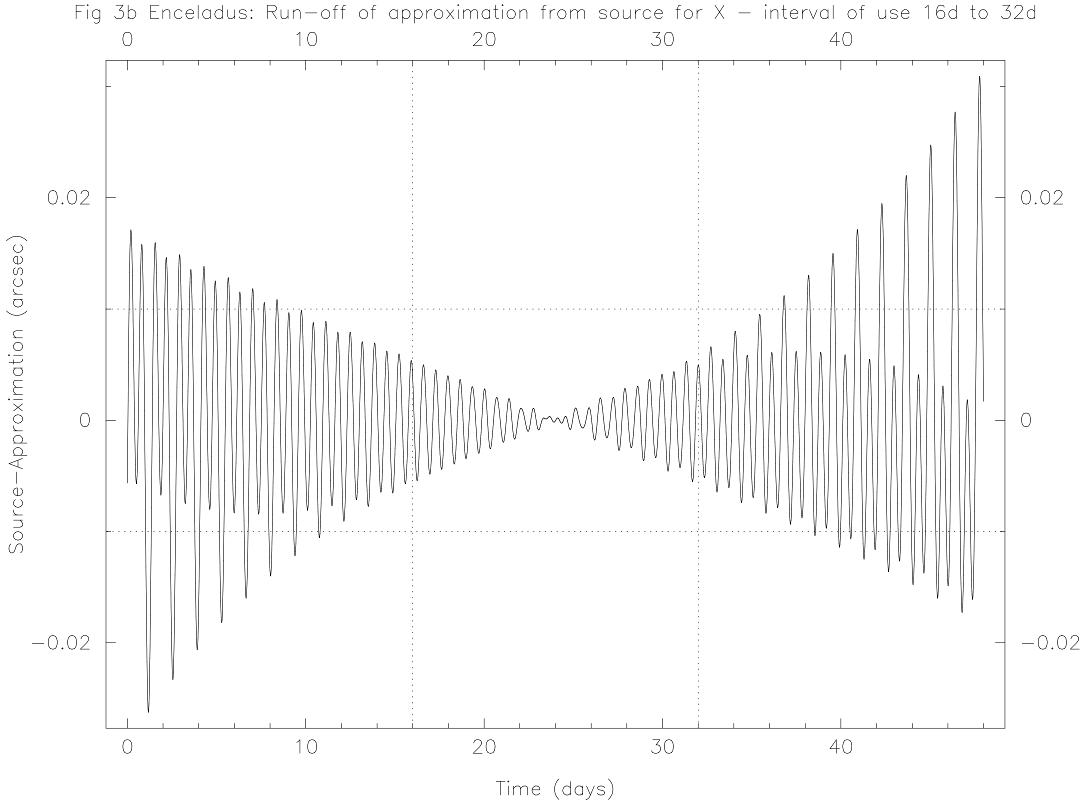


Fig 3c Tethys: Run−off of approximation from source for X − interval of use 16d to 32d

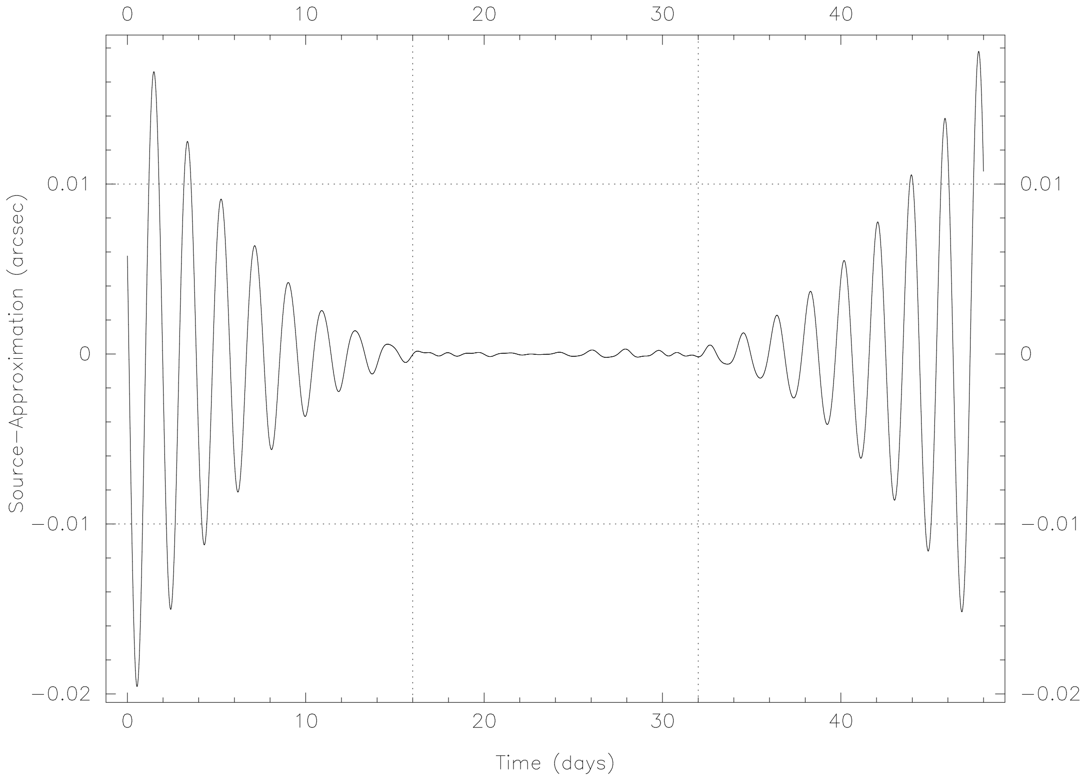
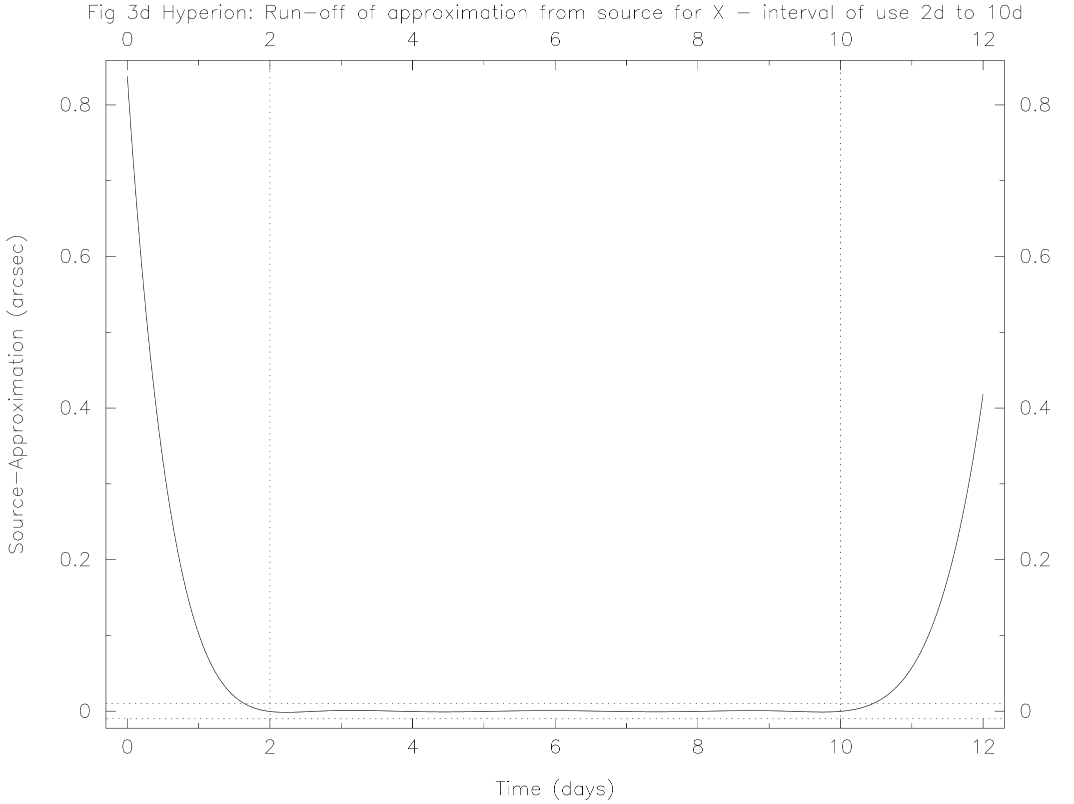


Fig 3d Hyperion: Run−off of approximation from source for X − interval of use 2d to 10d



The original theories except for Hyperion have a precision of about $0\overset{''}{.}2$. For Hyperion the precision is about $0\overset{''}{.}5$. This is due in part to the quality of observations used to fit the constants but also to deficiencies in the theory. From a comparison over the year of the Chebyshev representation and mixed function representation with the source ephemerides all differences were found to be in absolute value less than about $0\overset{''}{.}01$. Fig. 3 shows examples of the run-off of the mixed functions approximation from the source ephemerides for X for the satellites Mimas, Enceladus, Tethys and Hyperion.

Sample checks of coordinates from the mixed function representation with those from the French ephemerides showed differences of up to $0\overset{''}{.}05$ for Mimas, Enceladus, Tethys, Dione, Rhea and Titan, up to $0\overset{''}{.}15$ for Iapetus and up to $0\overset{''}{.}85$ for Hyperion. The differences found for Hyperion show the discrepancy in the original theories used.

On comparing the methods, the mixed function representation gains considerably over the Chebyshev approximation for the inner satellites Mimas, Enceladus, Tethys, Dione and Rhea as the fitting interval (a few orbital periods) is much longer at no expense to a marked increase in the number of coefficients to be determined. For the outer satellites the length of the intervals are comparable but the mixed function representation requires fewer coefficients to be determined. Thus we can say the total number of coefficients necessary to describe one year of the motion is always smaller in the case of the mixed functions. As an example of how the ephemerides may be represented in *The Astronomical Almanac*, sample pages for the ephemerides of Mimas, Titan and Iapetus are given in Appendix D.

6. Conclusions

Ephemerides for the differential coordinates of Saturn's major satellites have been developed using the Chebyshev approximation and method based on periodic and secular functions. The Chebyshev coefficients are determined from a discrete least-squares fit whilst in the mixed functions method the coefficients are determined from a continuous least-squares fit. The mixed functions representation is a more suitable approximation to the quasi-periodic character of the coordinates than the representation using Chebyshev polynomials. For fast-moving satellites the time interval for fitting over is much longer for the mixed function method than the Chebyshev approximation. For satellites with large orbital periods, like for example Iapetus, the gain is less marked. The mixed functions representation is easy to evaluate and provides an efficient way to store satellite differential coordinates and to keep the precision of the original theory. Thus, the quality of observations can be easily compared with precise ephemerides. For satellites with very long orbital periods(> 100d), the ephemerides would be more suitably stored using the Chebyshev approximation.

The programs written for computing the coefficients to represent the coordinates of the satellites of Saturn can easily be used to produce the coefficients of satellites in other planetary systems.

The programs are self-contained and do not use computer dependent utilities or external libraries and should be transferable to computers with FORTRAN compilers.

Table 3 Satellite Ephemerides for *The Astronomical Almanac*

Planet	Satellite	Source
Mars	Phobos, Deimos	Sinclair (1989)
Jupiter	Io, Europa, Ganymede, Callisto	Lieske (1994)
Saturn	Mimas, Enceladus, Tethys, Dione Rhea, Titan, Iapetus Hyperion	Harper and Taylor (1993) Taylor (1992)
Uranus	Ariel, Umbriel, Titania, Oberon, Miranda	Laskar and Jacobson (1987)
Neptune	Triton, Nereid	Jacobson *et al* (1991)
Pluto	Charon	Tholen (1990)

It is intended to redesign the satellite section of *The Astronomical Almanac* with the ephemerides of the major satellites displayed using the mixed functions representation. The proposed ephemerides to be used are given in Table 3 above. This list may be added to, to include some of the faint inner and outer satellites of the outer planets.

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A poster paper describing how compact ephemerides can be produced for Saturn's satellites but omitting the technical details, was presented at the *IAU Symposium No. 172* on *"Dynamics, Ephemerides and Astrometry in the Solar System"* , held in Paris, July 3-8, 1995.

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Appendix A

Expressions for the non-zero elements of matrix M

$$m_{1,1} = \int_{-1}^{+1} dx = 2$$

$$m_{1,2} = \int_{-1}^{1} \cos \omega x \;\; dx = \frac{2 \sin \omega}{\omega}$$

$$m_{1,3} = \int_{-1}^{+1} \cos 2\omega x \;\; dx = \frac{\sin 2\omega}{\omega}$$

$$m_{1,4} = m_{6,7} = \int_{-1}^{+1} x \sin \omega x \;\; dx = \frac{-2 \cos \omega}{\omega} + \frac{2 \sin \omega}{\omega^2}$$

$$m_{1,5} = m_{6,9} = \int_{-1}^{+1} x^2 \cos \omega x \;\; dx = \frac{2 \sin \omega}{\omega} + \frac{4 \cos \omega}{\omega^2} - \frac{4 \sin \omega}{\omega^3}$$

$$m_{2,2} = \int_{-1}^{+1} \cos^2 \omega x \;\; dx = 1 + \frac{\sin 2\omega}{2\omega}$$

$$m_{2,3} = \int_{-1}^{+1} \cos \omega x \cos 2\omega x \;\; dx = \frac{\sin 3\omega}{3\omega} + \frac{\sin \omega}{\omega}$$

$$m_{2,4} = m_{7,9} = \int_{-1}^{+1} x \cos \omega x \sin \omega x \;\; dx = \frac{-\cos 2\omega}{2\omega} + \frac{\sin 2\omega}{4\omega^2}$$

$$m_{2,5} = m_{9,9} = \int_{-1}^{+1} x^2 \cos^2 \omega x \;\; dx = \frac{1}{3} + \frac{\sin 2\omega}{2\omega} + \frac{\cos 2\omega}{2\omega^2} - \frac{\sin 2\omega}{4\omega^3}$$

$$m_{3,3} = \int_{-1}^{+1} \cos^2 2\omega x \;\; dx = 1 + \frac{\sin 4\omega}{4\omega}$$

$$m_{3,4} = \int_{-1}^{+1} x \cos 2\omega x \sin \omega x \;\; dx = \frac{-\cos 3\omega}{3\omega} + \frac{\sin 3\omega}{9\omega^2} + \frac{\cos \omega}{\omega} - \frac{\sin \omega}{\omega^2}$$

$$m_{3,5} = \int_{-1}^{+1} x^2 \cos 2\omega x \cos \omega x \;\; dx = \frac{\sin 3\omega}{3\omega} + \frac{2 \cos 3\omega}{9\omega^2} - \frac{2 \sin 3\omega}{27\omega^3} + \frac{\sin \omega}{\omega} + \frac{2 \cos \omega}{\omega^2} - \frac{2 \sin \omega}{\omega^3}$$

$$m_{4,4} = m_{7,10} = \int_{-1}^{+1} x^2 \sin^2 \omega x \;\; dx = \frac{1}{3} - \frac{\sin 2\omega}{2\omega} - \frac{\cos 2\omega}{2\omega^2} + \frac{\sin 2\omega}{4\omega^3}$$

$$m_{4,5} = m_{9,10} = \int_{-1}^{+1} x^3 \sin \omega x \cos \omega x \;\; dx = \frac{-\cos 2\omega}{2\omega} + \frac{3 \sin 2\omega}{4\omega^2} + \frac{3 \cos 2\omega}{4\omega^3} - \frac{3 \sin 2\omega}{8\omega^4}$$

$$m_{5,5} = \int_{-1}^{+1} x^4 \cos^2 \omega x \;\; dx = \frac{1}{5} + \frac{\sin 2\omega}{2\omega} + \frac{\cos 2\omega}{\omega^2} - \frac{3 \sin 2\omega}{2\omega^3} - \frac{3 \cos 2\omega}{2\omega^4} + \frac{3 \sin 2\omega}{4\omega^5}$$

$$m_{6,6} = \int_{-1}^{+1} x^2 \;\; dx = \frac{2}{3}$$

$$m_{6,8} = \int_{-1}^{+1} x \sin 2\omega x \;\; dx = \frac{-\cos 2\omega}{\omega} + \frac{\sin 2\omega}{2\omega^2}$$

$$m_{6,10} = \int_{-1}^{+1} x^3 \sin \omega x \;\; dx = \frac{-2 \cos \omega}{\omega} + \frac{6 \sin \omega}{w^2} + \frac{12 \cos \omega}{\omega^3} - \frac{12 \sin \omega}{\omega^4}$$

$$m_{7,7} = \int_{-1}^{+1} \sin^2 \omega x \;\; dx = 1 - \frac{\sin 2\omega}{2\omega}$$

$$m_{7,8} = \int_{-1}^{+1} \sin \omega x \sin 2\omega x \;\; dx = \frac{\sin \omega}{\omega} - \frac{\sin 3\omega}{3\omega}$$

$$m_{8,8} = \int_{-1}^{+1} \sin^2 2\omega x \ \ dx = 1 - \frac{\sin 4\omega}{4\omega}$$

$$m_{8,9} = \int_{-1}^{+1} x \sin 2\omega x \cos \omega x \ \ dx = \frac{-\cos 3\omega}{3\omega} + \frac{\sin 3\omega}{9\omega^2} - \frac{\cos \omega}{\omega} + \frac{\sin \omega}{\omega^2}$$

$$m_{8,10} = \int_{-1}^{+1} x^2 \sin 2\omega x \sin \omega x \ \ dx = \frac{\sin \omega}{\omega} - \frac{\sin 3\omega}{3\omega} + \frac{2\cos \omega}{\omega^2} - \frac{2\cos 3\omega}{9\omega^2} - \frac{2\sin \omega}{\omega^3} + \frac{2\sin 3\omega}{27\omega^3}$$

$$m_{10,10} = \int_{-1}^{+1} x^4 \sin^2 \omega x \ \ dx = \frac{1}{5} - \frac{\sin 2\omega}{2\omega} - \frac{\cos 2\omega}{\omega^2} + \frac{3\sin 2\omega}{2\omega^3} + \frac{3\cos 2\omega}{2\omega^4} - \frac{3\sin 2\omega}{4\omega^5}$$

Appendix B

Computation of integrals by Gaussian quadrature

Integration formulae such as Simpson's rule or the trapezoidal rule are based upon the use of equally spaced points. It was observed by Gauss that formulas of greater accuracy with a fixed number of points could be obtained if both the abscissas and the coefficients in the quadrature formula were unrestricted.

Suppose we want to evaluate the integral

$$S = \int_{-1}^{+1} g(x) \ dx.$$

An approximation to S is sought of the form

$$S \approx \sum_{i=1}^{n} w_i g(x_i)$$

where the weights w_i $(i = 1, 2, \ldots, n)$ and the arguments x_i $(i = 1, 2, \ldots, n)$ are to be determined so as to obtain the best possible accuracy. A formula is said to be of best possible accuracy if it is exact for all polynomials of as high a degree as possible. Since there are altogether $2n$ arbitrary parameters, we can hope to obtain a formula which is exact for all polynomials up to and including degree $2n - 1$. Such formulas have been determined and are called Gaussian n-point formulae, where x_i is the i^{th} zero of the Legendre polynomial $P_n(x)$ and w_i is the corresponding weight

$$\frac{1}{w_i} = \frac{1}{2}(1 - x_i^2)(P_n^{'}(x_i))^2.$$

Legendre polynomials are evaluated from the recursive relation

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

with $P_0 = 1$; and $P_1 = x$.

The derivative of $P_n(x)$ is a Gegenbauer polynomial

$$P_n^{'}(x) = C_{n-1}^{\frac{3}{2}}(x).$$

Gegenbauer polynomials are evaluated using the recursive relation

$$(n+1)C_{n+1}^{\frac{3}{2}} = (2n+3)xC_n^{\frac{3}{2}}(x) - (n+2)C_{n-1}^{\frac{3}{2}}(x)$$

with $C_0^{\frac{3}{2}} = 1$; $C_1^{\frac{3}{2}} = 3x$.

The roots x_i of $P_n(x)$ can be computed using the Newton-Raphson method

$$x_i = \lim_{k \to \infty} x_k^{(i)}$$

where

$$x_{k+1}^{(i)} = x_k^{(i)} - \frac{P_n(x_k^{(i)})}{P_n^{'}(x_k^{(i)})}.$$

To start the iteration an approximation of the i^{th} zero of $P_n(x)$ is required. It is given by

$$x_i^{(n)} = \cos\theta_i^{(n)}$$

with

$$\theta_i^{(n)} = \frac{4i-1}{4n+2}\pi + \frac{1}{8n^2}\cot\frac{4i-1}{4n+2}\pi.$$

As an example of using the above formulae, arguments and weights for the Gaussian 40 point formula are given in Table 4.

Table 4 Gaussian 40 point formula

Index	Argument x_i	Weight w_i
1	0.99823770971056	0.00452127709853
2	0.99072623869946	0.01049828453115
3	0.97725994998377	0.01642105838191
4	0.95791681921379	0.02224584919417
5	0.93281280827868	0.02793700698002
6	0.90209880696887	0.03346019528255
7	0.86595950321226	0.03878216797447
8	0.82461223083331	0.04387090818567
9	0.77830565142652	0.04869580763507
10	0.72731825518993	0.05322784698394
11	0.67195668461418	0.05743976909939
12	0.61255388966798	0.06130624249293
13	0.54946712509513	0.06480401345660
14	0.48307580168618	0.06791204581523
15	0.41377920437160	0.07061164739129
16	0.34199409082576	0.07288658239580
17	0.26815218500725	0.07472316905797
18	0.19269758070137	0.07611036190063
19	0.11608407067526	0.07703981816425
20	0.03877241750605	0.07750594797842
21	-0.03877241750605	0.07750594797842
22	-0.11608407067526	0.07703981816425
23	-0.19269758070137	0.07611036190063
24	-0.26815218500725	0.07472316905797
25	-0.34199409082576	0.07288658239580
26	-0.41377920437160	0.07061164739129
27	-0.48307580168618	0.06791204581523
28	-0.54946712509513	0.06480401345660
29	-0.61255388966798	0.06130624249293
30	-0.67195668461418	0.05743976909939
31	-0.72731825518993	0.05322784698394
32	-0.77830565142652	0.04869580763507
33	-0.82461223083331	0.04387090818567
34	-0.86595950321226	0.03878216797447
35	-0.90209880696887	0.03346019528255
36	-0.93281280827868	0.02793700698002
37	-0.95791681921379	0.02224584919417
38	-0.97725994998377	0.01642105838191
39	-0.99072623869946	0.01049828453115
40	-0.99823770971056	0.00452127709853

Appendix C

Expressions for $a_0, a_1, b_1, \phi_1, b_2, \phi_2, b_3, \phi_3, b_4$ and ϕ_4 in terms of $q_1, q_2, \ldots, q_{10}, t_c$ and Δt.

We have

$$F(x) = q_1 + q_2 \cos \omega x + q_3 \cos 2\omega x + q_4 x \sin \omega x + q_5 x^2 \cos \omega x \\ + q_6 x + q_7 \sin \omega x + q_8 \sin 2\omega x + q_9 x \cos \omega x + q_{10} x^2 \sin \omega x \qquad (C.1)$$

with

$$x = \frac{2(t+t_c)}{\Delta t} - 1 \qquad (C.2)$$

and

$$\omega = \frac{\nu \Delta t}{2}. \qquad (C.3)$$

Substituting $(C.2)$ and $(C.3)$ into $(C.1)$ we obtain, where the origin is $t_0 + t_c$

$$F(t) = a_0 + a_1 t + b_1 \sin(\nu t + \phi_1) + b_2 \sin(2\nu t + \phi_2) + b_3 t \sin(\nu t + \phi_3) + b_4 t^2 \sin(\nu t + \phi_4)$$

where

$$a_0 = q_1 - (1 - \frac{2t_c}{\Delta t}) q_6$$
$$a_1 = \frac{2q_6}{\Delta t}$$
$$b_1 = (\alpha_1^2 + \alpha_2^2)^{1/2}$$
$$\phi_1 = \tan^{-1}(\alpha_1/\alpha_2) - \frac{\nu}{2}(\Delta t - 2t_c)$$

where

$$\alpha_1 = q_2 + \frac{2q_9 t_c}{\Delta t} - q_9 + \frac{4q_5 t_c^2}{\Delta t^2} - \frac{4q_5 t_c}{\Delta t} + q_5$$
$$\alpha_2 = q_7 + \frac{2q_4 t_c}{\Delta t} - q_4 + \frac{4q_{10} t_c^2}{\Delta t^2} - \frac{4q_{10} t_c}{\Delta t} + q_{10}$$
$$b_2 = ({q_3}^2 + {q_8}^2)^{\frac{1}{2}}$$
$$\phi_2 = \tan^{-1}(q_3/q_8) - \nu(\Delta t - 2t_c)$$
$$b_3 = ({\alpha_3}^2 + {\alpha_4}^2)^{1/2}$$
$$\phi_3 = \tan^{-1}(\alpha_3/\alpha_4) - \frac{\nu}{2}(\Delta t - 2t_c)$$

where

$$\alpha_3 = \frac{2q_9}{\Delta t} + \frac{8q_5 t_c}{\Delta t^2} - \frac{4q_5}{\Delta t}$$
$$\alpha_4 = \frac{2q_4}{\Delta t} + \frac{8q_{10} t_c}{\Delta t^2} - \frac{4q_{10}}{\Delta t}$$
$$b_4 = \frac{4}{\Delta t^2}(q_5^2 + q_{10}^2)^{1/2}$$
$$\phi_4 = \tan^{-1}(q_5/q_{10}) - \frac{\nu}{2}(\Delta t - 2t_c)$$